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Gentzen-Type Calculi for Involutive Quantales

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Completeness and cut-elimination theorems are proved for some Gentzen-type sequent calculi which are closely related to non-commutative involutive quantales.

KEY WORDS: involutive quantale; sequent calculus; cut-elimination; quantum logic.

1. INTRODUCTION

Algebraic structures for quantum mechanics, such as ortholattices and quantales, have been studied by many researchers. Logics corresponding to ortholattices and their neighbors are called *quantum logics* in Birkhoff and von Neumann's sense. A number of Gentzen-type sequent calculi for such standard quantum logics were investigated comprehensively (see. e.g. Nishimura, 1994; Takano, 1995).

Quantales were introduced by Mulvey in an attempt to cast light on the connections between C^*-algebras and quantum mechanics (Mulvey, 1986; Rosenthal, 1990). A quantale-besed (non-commutative logic-theoretic) approach to quantum mechanics was developed by Piazza (1995). It is known that (commutative versions of) quantales are one of the semantics of linear logic (Ishihara and Hiraishi, 2001; Larchey-Wendling and Galmiche, 2000; Yetter, 1990). A linear-logical understanding of quantum mechanics was established by Pratt (1993), and a *linear quantum logic* and other related quantum logics were proposed by Faggian and Sambin (1998).

Involutive quantales were introduced by Mulvey and Pelletier (1992) in order to quantize the calculus of relations by Hoare and He (1987). Some variations of involutive quantales, such as Gelfand quantales, von Neumann quantales and Hilbert quantales, have also been widely studied (see e.g. Mulvey and Pelletier, 2001, 2002; Pelletier and Rosický, 1997).

Quantum logics corresponding to involutive quantales and Gelfand quantales were proposed and studied by MacCaull (1997) for involutive quantales and by Allwein and MacCaull (2001) for Gelfand quantales. In (MacCaull, 1997), some complete Kripke-type semantics, a Gentzen-type sequent calculus and a

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relational proof system were given for such an involutive-quantale logic. The relationship between a kind of involutive-quantale logic, called *quantized intuitionistic linear logic* (QILL), and Wansing's extended intuitionistic linear logic with strong negation (Wansing, 1993) was also clarified by Kamide (2004). In (Kamide, 2004), a *commutative*-involutive-quantale model, a Kripke model and a number of Gentzen-type cut-free sequent calculi having the characteristic property of *quantization principle* were also given for QILL.

In the present paper, some *non-commutative* versions of involutive quantales are discussed along the lines of Kamide (2004). Some cut-free sequent calculi together with extended phase space models are given with respect to such non-commutative versions. Then, a logical understanding of the difference between involutive quantales and quantales can be obtained using these calculi and models. The calculi proposed are introduced as alternatives to the standard quantum logics.

The contents of this paper are then summarized as follows.

In Section 2, three new logics: basic involutive-quantale logic (BIQL), quasi-involutive-quantale (or twist-free-involutive-quantale) logic (QIQL) and involutive-quantale logic (IQL) are introduced as extensions of full Lambek logic (FL) or equivalently non-commutative intuitionistic linear logic, and the cut-elimination theorems for BIQL and QIQL are proved using a new embedding result. By assuming the exchange rule, the logics BIQL, QIQL and IQL are theorem-equivalent, i.e. theorem-equivalent to QILL in (Kamide, 2004). These syntactical investigations clarify that twist-free-involutive quantales, which correspond to QIQL, are essentially equivalent to quantales, which correspond to FL.

In Section 3, an involutive-quantale model for IQL and a twist-free-involutive-quantale model for QIQL are introduced, and the soundness theorems for IQL and QIQL (Theorem 3.5), and the completeness theorem (with respect to the twist-free-involutive-quantale model) for QIQL (Theorem 3.6) are addressed. The completeness theorem (with respect to the involutive-quantale model) for IQL is remained an open question.

In Section 4, Theorems 3.5 and 3.6 are proved along the lines of Kamide (2004).

In Section 5, extended intuitionistic non-commutative phase models are introduced for QIQL and BIQL, and the soundness and completeness theorems (Theorem 5.5) are addressed as a main result in this paper. In such extended models, the difference from the standard intuitionistic non-commutative phase model for FL is only to use a negative valuation v^{-}, which characterizes the involution operator appearing in involutive and twist-free-involutive quantales. This fact also means semantically that twist-free-involutive quantales are essentially equivalent to quantales.

In Section 6, Theorem 5.5 is proved using an extended version of the method by Okada (2002). This proof simultaneously derives the cut-elimination theorems for QIQL and BIQL.

Prior to the detailed discussion, the language and notion used in this paper are introduced later. *Formulae* are constructed from propositional variables, propositional constants $\mathbf{1}$, $\top$, and $\bot$, $\rightarrow$ (implication), $\leftarrow$ (left implication), $\wedge$ (conjunction), $*$ (fusion), $\vee$ (disjunction) and $\cdot^{\bullet}$ (involution). Small letters $p, q, \ldots$ are used to denote propositional variables, Greek small letters $\alpha, \beta, \ldots$ are used to denote formulae, and Greek capital letters $\Gamma, \Delta, \ldots$ are used to represent finite (possibly empty) sequences of formulae. $\Gamma^{\bullet}$ denotes the sequence $\langle \gamma^{\bullet} | \gamma \in \Gamma \rangle$. A sequence Γ is also expressed as $[\Gamma]$. A *sequent* is an expression of the form $\Gamma \Rightarrow \alpha$ where α is non-empty (i.e. a formula). The symbol $\equiv$ is used to denote equality as sequences of symbols. Δ^* denotes $\delta_1 * \cdots * \delta_n$ if $\Delta \equiv \langle \delta_1, \ldots, \delta_n \rangle$ $(1 \leq n)$, and denotes an empty sequence if Δ is empty. $\Delta^{\star}$ denotes Δ^* if $\Delta \equiv \langle \delta_1, \ldots, \delta_n \rangle$ $(1 \leq n)$, and denotes $\mathbf{1}$ if Δ is empty. If a sequent S is provable in a sequent system L, then such a fact is denoted as $L \vdash S$, and sometimes denoted as $\vdash S$ for $L \vdash S$ by omitting L. Since all logics discussed in this paper are formulated as sequent calculi, we will occasionally identify a sequent calculus with the logic determined by it.

2. SEQUENT CALCULI

First, we introduce FL (full Lambek logic[2]).

The initial sequents of FL are of the forms:

$$\alpha \Rightarrow \alpha, \quad \Rightarrow \mathbf{1}, \quad \Gamma \Rightarrow \top, \quad \Gamma, \bot, \Delta \Rightarrow \gamma.$$

The cut rule of FL is of the form:

$$\frac{\Delta \Rightarrow \alpha \quad \Gamma, \alpha, \Sigma \Rightarrow \gamma}{\Gamma, \Delta, \Sigma \Rightarrow \gamma}\text{(cut)}.$$

The inference rules of FL are of the forms:

$$\frac{\Gamma, \Delta \Rightarrow \gamma}{\Gamma, \mathbf{1}, \Delta \Rightarrow \gamma}(\mathbf{1}\text{we}),$$

$$\frac{\Delta \Rightarrow \alpha \quad \Gamma, \beta, \Sigma \Rightarrow \gamma}{\Gamma, \alpha \rightarrow \beta, \Delta, \Sigma \Rightarrow \gamma}(\rightarrow\text{left}), \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta}(\rightarrow\text{right}),$$

$$\frac{\Delta \Rightarrow \alpha \quad \Gamma, \beta, \Sigma \Rightarrow \gamma}{\Gamma, \Delta, \alpha \leftarrow \beta, \Sigma \Rightarrow \gamma}(\leftarrow\text{left}), \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \leftarrow \beta}(\leftarrow\text{right}),$$

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \gamma}{\Gamma, \alpha * \beta, \Delta \Rightarrow \gamma}(*\text{left}), \qquad \frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha * \beta}(*\text{right}),$$

[2] Strictly speaking, the logic presented is the propositional full Lambek logic without the multiplicative falsum constant **0**, or equivalently the modality-free propositional non-commutative intuitionistic linear logic without **0**.

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \gamma}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \gamma}(\wedge\,\text{left1}), \quad \frac{\Gamma, \beta, \Delta \Rightarrow \gamma}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \gamma}(\wedge\,\text{left2}), \quad \frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}(\wedge\,\text{right}),$$

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \gamma \quad \Gamma, \beta, \Delta \Rightarrow \gamma}{\Gamma, \alpha \vee \beta, \Delta \Rightarrow \gamma}(\vee\,\text{left}), \quad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}(\vee\,\text{right1}), \quad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}(\vee\,\text{right2}).$$

BIQL (basic involutive-quantale logic) is obtained from FL by adding the initial sequents and inference rules of the forms:

$$\Rightarrow \mathbf{1}^{\bullet}, \quad \Gamma \Rightarrow \top^{\bullet}, \quad \Gamma, \bot^{\bullet}, \Delta \Rightarrow \gamma,$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha^{\bullet\bullet}}(\bullet\,\text{right}), \qquad \frac{\Gamma, \alpha, \Delta \Rightarrow \gamma}{\Gamma, \alpha^{\bullet\bullet}, \Delta \Rightarrow \gamma}(\bullet\text{left}),$$

$$\frac{\Gamma, \Delta \Rightarrow \gamma}{\Gamma, \mathbf{1}^{\bullet}, \Delta \Rightarrow \gamma}(\bullet\mathbf{1}\,\text{we}),$$

$$\frac{\Delta \Rightarrow \alpha^{\bullet} \quad \Gamma, \beta^{\bullet}, \Sigma \Rightarrow \gamma}{\Gamma, (\alpha \rightarrow \beta)^{\bullet}, \Delta, \Sigma \Rightarrow \gamma}(\bullet \rightarrow \text{left}), \quad \frac{\Gamma, \alpha^{\bullet} \Rightarrow \beta^{\bullet}}{\Gamma \Rightarrow (\alpha \rightarrow \beta)^{\bullet}}(\bullet \rightarrow \text{right}),$$

$$\frac{\Delta \Rightarrow \alpha^{\bullet} \quad \Gamma, \beta^{\bullet}, \Sigma \Rightarrow \gamma}{\Gamma, \Delta, (\alpha \leftarrow \beta)^{\bullet}, \Sigma \Rightarrow \gamma}(\bullet \leftarrow \text{left}), \quad \frac{\alpha^{\bullet}, \Gamma \Rightarrow \beta^{\bullet}}{\Gamma \Rightarrow (\alpha \leftarrow \beta)^{\bullet}}(\bullet \leftarrow \text{right}),$$

$$\frac{\Gamma, \beta^{\bullet}, \alpha^{\bullet}, \Delta \Rightarrow \gamma}{\Gamma, (\alpha * \beta)^{\bullet}, \Delta \Rightarrow \gamma}(\bullet * \text{left}), \quad \frac{\Gamma \Rightarrow \beta^{\bullet} \quad \Delta \Rightarrow \alpha^{\bullet}}{\Gamma, \Delta \Rightarrow (\alpha * \beta)^{\bullet}}(\bullet * \text{right}),$$

$$\frac{\Gamma, \alpha^{\bullet}, \Delta \Rightarrow \gamma}{\Gamma, (\alpha \wedge \beta)^{\bullet}, \Delta \Rightarrow \gamma}(\bullet \wedge \text{left 1}), \quad \frac{\Gamma, \beta^{\bullet}, \Delta \Rightarrow \gamma}{\Gamma, (\alpha \wedge \beta)^{\bullet}, \Delta \Rightarrow \gamma}(\bullet \wedge \text{left 2}),$$

$$\frac{\Gamma \Rightarrow \alpha^{\bullet} \quad \Gamma \Rightarrow \beta^{\bullet}}{\Gamma \Rightarrow (\alpha \wedge \beta)^{\bullet}}(\bullet \wedge \text{right}), \quad \frac{\Gamma, \alpha^{\bullet}, \Delta \Rightarrow \gamma \quad \Gamma, \beta^{\bullet}, \Delta \Rightarrow \gamma}{\Gamma, (\alpha \vee \beta)^{\bullet}, \Delta \Rightarrow \gamma}(\bullet \vee \text{left}),$$

$$\frac{\Gamma \Rightarrow \alpha^{\bullet}}{\Gamma \Rightarrow (\alpha \vee \beta)^{\bullet}}(\bullet \vee \text{right 1}), \quad \frac{\Gamma \Rightarrow \beta^{\bullet}}{\Gamma \Rightarrow (\alpha \vee \beta)^{\bullet}}(\bullet \vee \text{right 2}).$$

We define two logics IQL (involutive-quantale logic) and QIQL (quasi-involutive- or twist-free-involutive-quantale logic) later.

$$\text{IQL} = \text{BIQL} + (\bullet\text{mono}) + (\bullet\text{mono}^{-1})$$

where ($\bullet$mono) and ($\bullet$mono^{-1}) are of the forms:

$$\frac{\alpha \Rightarrow \beta}{\alpha^{\bullet} \Rightarrow \beta^{\bullet}}(\bullet\,\text{mono}), \qquad \frac{\alpha^{\bullet} \Rightarrow \beta^{\bullet}}{\alpha \Rightarrow \beta}(\bullet\,\text{mono}^{-1})$$

where α may be empty.

$$\text{QIQL} = \text{BIQL} - (\bullet * \text{left}) - (\bullet * \text{right}) + (\bullet * \text{left}') + (\bullet * \text{right}')$$

where ($\bullet*$left$'$) and ($\bullet*$right$'$) are of the forms:

$$\frac{\Gamma, \alpha^\bullet, \beta^\bullet, \Delta \Rightarrow \gamma}{\Gamma, (\alpha * \beta)^\bullet, \Delta \Rightarrow \gamma}(\bullet * \text{ left}'), \qquad \frac{\Gamma \Rightarrow \alpha^\bullet \quad \Delta \Rightarrow \beta^\bullet}{\Gamma, \Delta \Rightarrow (\alpha * \beta)^\bullet}(\bullet * \text{ right}').$$

Next, we give two embeddings of BIQL and QIQL into FL, which are a slight modification of the embedding (for a logic with strong negation) introduced by Rautenberg (1979). We fix a set *PR* of propositional variables used as components of the language of the logics with $\cdot^\bullet$, and define the set $PR' := \{p' | p \in PR\}$ of propositional variables. The language $\mathcal{L}^\bullet$ of the logics with $\cdot^\bullet$ is defined by using PR, $\mathbf{1}$, $\top$, $\bot$, $\wedge$, $\vee$, $*$, $\rightarrow$, $\leftarrow$ and $\cdot^\bullet$. The language $\mathcal{L}$ of FL is obtained from $\mathcal{L}^\bullet$ by adding PR' and by deleting $\cdot^\bullet$.

Definition 2.1. A mapping f from $\mathcal{L}^\bullet$ to $\mathcal{L}$ is defined as follows.

1. $f(p) := p$ and $f(p^\bullet) := p' \in PR'$ for any $p \in PR$,
2. $f(\diamondsuit) := \diamondsuit$ where $\diamondsuit \in \{\mathbf{1}, \top, \bot\}$,
3. $f(\alpha\diamondsuit\beta) := f(\alpha)\diamondsuit f(\beta)$ where $\diamondsuit \in \{*, \wedge, \vee, \rightarrow, \leftarrow\}$,
4. $f(\diamondsuit^\bullet) := \diamondsuit$ where $\diamondsuit \in \{\mathbf{1}, \top, \bot\}$,
5. $f(\alpha^{\bullet\bullet}) := f(\alpha)$,
6. $f((\alpha * \beta)^\bullet) := f(\beta^\bullet) * f(\alpha^\bullet)$,
7. $f((\alpha\diamondsuit\beta)^\bullet) := f(\alpha^\bullet)\diamondsuit f(\beta^\bullet)$ where $\diamondsuit \in \{\wedge, \vee, \rightarrow, \leftarrow\}$.
   A mapping g from $\mathcal{L}^\bullet$ to $\mathcal{L}$ is also defined as the same conditions 1–5 and 7, and the following condition.
8. $g((\alpha * \beta)^\bullet) := g(\alpha^\bullet) * g(\beta^\bullet)$.

Let Γ be a sequence of formulae in $\mathcal{L}^\bullet$. Then, $f(\Gamma)$ (or $g(\Gamma)$) denotes the result of replacing every occurence of a formula α in Γ by an occurence of $f(\alpha)$ (or $g(\alpha)$, respectively). The following proposition means that QIQL is essentially equivalent to FL, i.e. the involution operator can be expressed as propositional variables. This means syntactically that twist-free-involutive quantales, which correspnd to QIQL, are essentially equivalent to quantales, which correspond to FL.

Proposition 2.2. (Involution-elimination) *Let Γ be a sequence of formulae in $\mathcal{L}^\bullet$, γ be a formula in $\mathcal{L}^\bullet$, and f and g be mappings defined in Definition 2.1.*

(1) *if* BIQL $\vdash \Gamma \Rightarrow \gamma$, *then* FL $\vdash f(\Gamma) \Rightarrow f(\gamma)$.
(2) *if* FL $-$ (cut) $\vdash f(\Gamma) \Rightarrow f(\gamma)$, *then* BIQL $-$ (cut) $\vdash \Gamma \Rightarrow \gamma$,
(3) *if* QIQL $\vdash \Gamma \Rightarrow \gamma$, *then* FL $\vdash g(\Gamma) \Rightarrow g(\gamma)$.
(4) *if* FL $-$ (cut) $\vdash g(\Gamma) \Rightarrow g(\gamma)$, *then* QIQL $-$ (cut) $\vdash \Gamma \Rightarrow \gamma$.

We may not derive such an resemble result for IQL directly because of the existence of (•mono).

Using Proposition 2.2, we can show the following main theorem.

Theorem 2.3. (Cut-elimination for BIQL and QIQL) *Let L be* BIQL *or* QIQL. *The rule* (cut) *is admissible in cut-free L.*

Proof: We only show the case for BIQL. Suppose that BIQL $\vdash \Gamma \Rightarrow \gamma$. Then, we have FL $\vdash f(\Gamma) \Rightarrow f(\gamma)$ by Proposition 2.2 (1). Assuming the well-known cut-elimination theorem for FL, we can obtain FL$-$(cut) $\vdash f(\Gamma) \Rightarrow f(\gamma)$. By Proposition 2.2 (2), we obtain BIQL$-$(cut) $\vdash \Gamma \Rightarrow \gamma$. □

This theorem will also be proved semantically in Section 6.

We do not know whether the cut-elimination theorem for IQL holds or not.

Using Theorem 2.3, we can show the following.[3]

Corollary 2.4. *Let L be* BIQL *and* QIQL. *L is decidable and is a conservative extension of* FL.

We remark that the rules of the forms:

$$\frac{\Gamma \Rightarrow \alpha^{\bullet\bullet}}{\Gamma \Rightarrow \alpha}(\bullet\,\text{right}^{-1}), \qquad \frac{\Gamma, \alpha^{\bullet\bullet}, \Delta \Rightarrow \gamma}{\Gamma, \alpha, \Delta \Rightarrow \gamma}(\bullet\,\text{left}^{-1})$$

are admissible in *cut-free* BIQL and *cut-free* QIQL, and derivable in IQL.

We then have the following.[4]

Lemma 2.5. *The rule*

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma^{\bullet} \Rightarrow \alpha^{\bullet}}(\bullet\,\text{regu})$$

is admissible in cut-free QIQL.

Proof: We prove this by induction on the cut-free proof P of the upper sequent $\Gamma \Rightarrow \alpha$ of (•regu) in QIQL. We distinguish the cases according to the last inference of P. We show only the following case.

(Case (•∗left′)): The last inference rule of P is of the form:

$$\frac{\Gamma_1, \beta^{\bullet}, \gamma^{\bullet}, \Gamma_2 \Rightarrow \alpha}{\Gamma_1, (\beta * \gamma)^{\bullet}, \Gamma_2 \Rightarrow \alpha}(\bullet * \text{left}')$$

[3] BIQL and QIQL have no subformula property, but we can give the calculi called the "subformula calculi" which have such a property, by applying a similar way as in (Kamide, 2004).

[4] An analogous result for a negation rule holds for a minimal quantum logic (see Takano, 1995).

where $\Gamma \equiv \langle \Gamma_1, (\beta * \gamma)^\bullet, \Gamma_2 \rangle$. By the hypothesis of induction, we have that $\vdash \Gamma_1^\bullet, \beta^{\bullet\bullet}, \gamma^{\bullet\bullet}, \Gamma_2^\bullet \Rightarrow \alpha^\bullet$, and hence

$$\dfrac{\dfrac{\begin{array}{c}\Gamma_1^\bullet, \beta^{\bullet\bullet}, \gamma^{\bullet\bullet}, \Gamma_2^\bullet \Rightarrow \alpha^\bullet \\ \vdots (\bullet\text{left}^{-1}) \\ \Gamma_1^\bullet, \beta, \gamma, \Gamma_2^\bullet \Rightarrow \alpha^\bullet\end{array}}{\Gamma_1^\bullet, \beta * \gamma, \Gamma_2^\bullet \Rightarrow \alpha^\bullet}(*\text{left})}{\Gamma_1^\bullet, (\beta * \gamma)^{\bullet\bullet}, \Gamma_2^\bullet \Rightarrow \alpha^\bullet}(\bullet\text{left}).$$

□

We remark that this lemma does not work for BIQL because the application of $(*\text{left})$ in the proof of the case for $(\bullet*\text{left})$ in a similar setting displayed earlier cannot be adopted.

The rule

$$\frac{\Gamma^\bullet \Rightarrow \alpha^\bullet}{\Gamma \Rightarrow \alpha}(\bullet\text{regu}^{-1})$$

is also derivable in QIQL + ($\bullet$regu).

Then, we have the following theorem.

Theorem 2.6. QIQL *and* QIQL + ($\bullet$regu) + ($\bullet$regu^{-1}) *are theorem-equivalent.*

We need this theorem to prove the completeness theorem (w.r.t. twist-free-involutive-quantale model) for QIQL.

Next, we consider the exchange rule:

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \gamma}{\Gamma, \beta, \alpha, \Delta \Rightarrow \gamma}(\text{ex}).$$

Then, we have the following.

Proposition 2.7. BIQL + (ex)*,* QIQL + (ex) *and* IQL + (ex) *are theorem-equivalent.*

We note that QIQL + (ex) is theorem-equivalent to QILL in (Kamide, 2004).

Using ($\bullet$regu), ($\bullet$regu^{-1}), ($\bullet$mono) and ($\bullet$mono^{-1}), we can obtain the following characteristic property which is introduced in (Kamide, 2004).

Theorem 2.8. (Quantization Principle) *Let L be* QIQL *or* IQL*. For any formula* α*,* $L \vdash\ \Rightarrow \alpha$ *if and only if* $L \vdash\ \Rightarrow \alpha^\bullet$*.*

This theorem means intuitively that the existence of the parallel worlds in the sense of the theory of quantum mechanics, i.e. there are a number of worlds (including our real world) with coherence in parallel. In this context, "$\vdash\ \Rightarrow \alpha$" means "$\alpha$ is true in our (chosen) real-world", and "$\vdash\ \Rightarrow \alpha^{\bullet}$" means "$\alpha$ is true in another (unchosen) parallel world."

Finally in this section, we review the original involutive-quantale logic, called FL_I, which is introduced by MacCaull (1997). FL_I is obtained from FL (with the addition of the inference rule and initial sequent for the multiplicative falsum constant $\mathbf{0}$) by adding the following inference rules:

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma^{*\bullet} \Rightarrow \alpha^{\bullet}}, \quad \frac{\Gamma, \gamma \Rightarrow \alpha}{\Gamma, \gamma \Rightarrow \alpha^{\bullet\bullet}}, \quad \frac{\Gamma, \gamma \Rightarrow \alpha^{\bullet\bullet}}{\Gamma, \gamma \Rightarrow \alpha}$$

$$\frac{\Gamma, \gamma \Rightarrow \alpha^{\bullet} * \beta^{\bullet}}{\Gamma, \gamma \Rightarrow (\beta * \alpha)^{\bullet}}, \quad \frac{\Gamma, \gamma \Rightarrow (\alpha * \beta)^{\bullet}}{\Gamma, \gamma \Rightarrow (\beta^{\bullet} * \alpha)^{\bullet}}, \quad \frac{\Gamma, \gamma \Rightarrow \bigvee(\alpha_i^{\bullet})}{\Gamma, \gamma \Rightarrow (\bigvee \alpha_i)^{\bullet}}, \quad \frac{\Gamma, \gamma \Rightarrow (\bigvee \alpha_i)^{\bullet}}{\Gamma, \gamma \Rightarrow \bigvee(\alpha_i^{\bullet})}$$

where $\vee$ is an infinite disjunction connective.

3. INVOLUTIVE QUANTALE MODELS

Definition 3.1. (Quantale) A *unitale quantale* is a structure $\mathbf{Q} := \langle \mathbf{Q}, \bigcup, \cdot, \dot{\mathbf{1}} \rangle$ satisfying the following conditions:

1. $\langle Q, \bigcup \rangle$ is a complete lattice (the least element and the greatest element are respectively denoted by $\dot{\bot}$ and $\dot{\top}$, and the binary versions of the lattice operations are denoted by $\cup$ and $\cap$),
2. $\langle Q, \cdot, \dot{\mathbf{1}} \rangle$ is a monoid with the identity $\dot{\mathbf{1}}$,
3. $(\bigcup x_i) \cdot y = \bigcup(x_i \cdot y)$ and $y \cdot (\bigcup x_i) = \bigcup(y \cdot x_i)$ for all $x_i, y \in Q$.

We define two operations $\dot{\rightarrow}$ and $\dot{\leftarrow}$ on Q as follows:

$$y \dot{\rightarrow} z := \bigcup\{x | x \cdot y \le z\} \quad \text{and} \quad y \dot{\leftarrow} z := \bigcup\{x | y \cdot x \le z\}$$

where $\le$ is defined as $x \le y$ iff $x \cup y = y$ for all $x, y \in Q$. Then the following condition on Q holds using the condition 3 mentioned earlier:

$$(x \le y \dot{\rightarrow} z \ \text{ iff } \ x \cdot y \le z) \ \text{ and } \ (x \le y \dot{\leftarrow} z \ \text{ iff } \ y \cdot x \le z) \ \text{ for all } \ x, y, z \in Q.$$

We call the unitale quantale equiped with $\dot{\bot}$, $\dot{\top}$, $\cup$, $\cap$, $\dot{\rightarrow}$ and $\dot{\leftarrow}$, quantale in the following.

We remark that the following monotonicity condition on a quantale $\mathbf{Q}$ holds:

$x \le x'$ and $y \le y'$ imply $x \cdot y \le x' \cdot y'$, $x' \dot{\rightarrow} y \le x \dot{\rightarrow} y'$ and $x' \dot{\leftarrow} y \le x \dot{\leftarrow} y'$ for all $x, x', y, y' \in Q$.

Definition 3.2. (Involutive and twist-free-involutive quantales) An *involutive quantale* is a structure $\mathbf{Q}^{\bullet} := \langle \mathbf{Q}, \cdot^{\circ} \rangle$ satisfying the following conditions:

1. **Q** is a quantale $\langle Q, \bigcup, \cdot, \dot{\mathbf{1}}\rangle$ equiped with $\dot{\bot}, \dot{\top}, \cup, \cap, \dot{\rightarrow}$ and $\dot{\leftarrow}$ (Definition 3.1),
2. $\cdot^{\circ}$ is a unary operation on Q such that

   C1: $x^{\circ\circ} = x$,
   C2: $(\bigcup x_i)^{\circ} = \bigcup (x_i)^{\circ}$,
   C3: $(x \cdot y)^{\circ} = y^{\circ} \cdot x^{\circ}$(*twist condition*),
   C4: $(x \cap y)^{\circ} = x^{\circ} \cap y^{\circ}$,
   C5: $(x \dot{\rightarrow} y)^{\circ} = x^{\circ} \dot{\rightarrow} y^{\circ}$,
   C6: $(x \dot{\leftarrow} y)^{\circ} = x^{\circ} \dot{\leftarrow} y^{\circ}$,
   C7: $\dot{\mathbf{1}}^{\circ} = \dot{\mathbf{1}}$,
   C8: $\dot{\top}^{\circ} = \dot{\top}$,
   C9: $\dot{\bot}^{\circ} = \dot{\bot}$.

A *twist-free-involutive* or *quasi-involutive quantale* is a structure $\mathbf{Q}^{\star} := \langle \mathbf{Q}, \cdot^{\circ}\rangle$ satisfying the same condition 1 earlier, and $\cdot^{\circ}$ is a unary operation on **Q** satisfying the same conditions C1, C2 and C4−C9 earlier, and the following condition[5]:

$$\text{C10} : (x \cdot y)^{\circ} = x^{\circ} \cdot y^{\circ}.$$

We can derive the following condition on $\mathbf{Q}^{\bullet}$ and $\mathbf{Q}^{\star}$ by using C1 and C2:

$$\text{C2}' : x \leq y \text{ iff } x^{\circ} \leq y^{\circ} \text{ for all } x, y \in Q.$$

The original involutive quantales in (Mulvey and Pelletier, 1992) do not have the conditions C4, C5, C6, C8, C9, C10 (and the operations and constants $\cap, \dot{\rightarrow}, \dot{\leftarrow}, \dot{\bot}, \dot{\top}$).

Definition 3.3. A *valuation* v on an involutive quantale $\mathbf{Q}^{\bullet}$ is a mapping from the set of all propositional variables to Q. A valuation v is extended to a mapping from the set of all formulae to Q by

1. $v(\mathbf{1}) := \dot{\mathbf{1}}$,
2. $v(\top) := \dot{\top}$,
3. $v(\bot) := \dot{\bot}$,
4. $v(\alpha \wedge \beta) := v(\alpha) \cap v(\beta)$,
5. $v(\alpha \vee \beta) := v(\alpha) \cup v(\beta)$,
6. $v(\alpha * \beta) := v(\alpha) \cdot v(\beta)$,
7. $v(\alpha \rightarrow \beta) := v(\alpha) \dot{\rightarrow} v(\beta)$,
8. $v(\alpha \leftarrow \beta) := v(\alpha) \dot{\leftarrow} v(\beta)$,
9. $v(\alpha^{\bullet}) := v(\alpha)^{\circ}$.

A valuation v on a twist-free-involutive quantale $\mathbf{Q}^{\star}$ is the same as that for $\mathbf{Q}^{\bullet}$.

[5] We remark that the conditions C3 and C10, respectively, correspond to the pair {(•∗left), (•∗right)} and {(•∗left′), (•∗right′)}.

Definition 3.4. (Involutive and twist-free-involutive quantale models) An *involutive quantale model* is a structure $\langle \mathbf{Q}^\bullet, v\rangle$ such that $\mathbf{Q}^\bullet$ is an involutive quantale and v is a valuation on $\mathbf{Q}^\bullet$. A formula α is *true* in an involutive quantale model $\langle \mathbf{Q}^\bullet, v\rangle$ if $\dot{\mathbf{1}} \leq v(\alpha)$ holds, and *valid* in an involutive quantale $\mathbf{Q}^\bullet$ if it is true for any valuation v on the involutive quantale. A sequent $\alpha_1, \ldots, \alpha_n \Rightarrow \beta$ (or $\Rightarrow \beta$) is *true* in an involutive quantale model $\langle \mathbf{Q}^\bullet, v\rangle$ if the formula $\alpha_1 * \cdots * \alpha_n \rightarrow \beta$ (or β) is true in it, and valid in an involutive quantale if so is $\alpha_1 * \cdots * \alpha_n \rightarrow \beta$ (or β). A twist-free-involutive quantale model $\langle \mathbf{Q}^\star, v\rangle$ and the corresponding notions as defined earlier are also defined similarly.

By using a similar (but a slightly different) way as in (Kamide, 2004), we can show the following soundness and completeness theorems. The proofs of these theorems will be given in the next section.

Theorem 3.5. (Soundness for IQL and QIQL) *Let C_1 be the class of all involutive quantales, C_2 be the class of all twist-free-involutive quantales, $L(C_1) :=$ $\{S \mid$ a sequent S is valid in all involutive quantales of $C_1\}$, $L(C_2) := \{S \mid$ a sequent S is valid in all twist-free-involutive quantales of $C_2\}$, $L_1 := \{S | \text{IQL} \vdash S\}$ and $L_2 := \{S | \text{QIQL} \vdash S\}$. Then, $L_1 \subseteq L(C_1)$ and $L_2 \subseteq L(C_2)$.*

In the proof of this theorem, we have to use the operation $\leftarrow$ for the case of ($\bullet\!\!\rightarrow$left) in the induction step. We can not adopt ($\bullet$regu) to IQL, because the twist condition derives the fact that $v(\gamma_1 * \gamma_2 * \gamma_3)^\circ = v(\gamma_3)^\circ \cdot v(\gamma_2)^\circ \cdot v(\gamma_1)^\circ$.

Theorem 3.6. (Completeness for QIQL) *Let $L(C_2)$ and L_2 be the same as that in Theorem 3.5. Then, $L(C_2) \subseteq L_2$.*

This theorem is proved for QIQL + ($\bullet$regu) + ($\bullet$regu^{-1}), which is theorem-equivalent to QIQL by Theorem 2.6, constructing a canonical twist-free-involutive-quantale model by using MacNeill completion technique. The constraction can be obtained based on (Ishihara and Hiraishi, 2001; Kamide, 2004). The main differences from the proofs for the non-modal intuitionistic linear logic in (Ishihara and Hiraishi, 2001; Larchey-Wendling and Galmiche, 2000) are the exsistence of the case for $\cdot^\bullet$ and the loss of the commutativity for the monoid operation.

We may not prove the same theorem for IQL or BIQL, because we must use ($\bullet$regu) and ($\bullet$regu^{-1}), and the facts $\vdash \Gamma^{\bullet\star} \Rightarrow \Gamma^{\star\bullet}$ and $\vdash \Gamma^{\star\bullet} \Rightarrow \Gamma^{\bullet\star}$, which are not compatible to the twist condition.

The following is thus remained an open question.

Question: Is IQL complete with respect to the presented involutive quantale model?

4. PROOFS OF THEOREMS 3.5 AND 3.6

4.1. Proof of Theorem 3.5

We only show the proof for the theorem for QIQL by induction on a proof P of QIQL. The proof is straightforward and similar to that for FL. We distinguish the cases according to the last inference rules in P. We assume the associativity for the monoid operation $\cdot$, and hence we do not use the parenthesis with respect to $\cdot$.

(Case ($\bullet\!\rightarrow$left)):[6] The last inference rule of P is of the form:

$$\frac{\Delta \Rightarrow \alpha^{\bullet} \quad \Gamma, \beta^{\bullet}, \Sigma \Rightarrow \gamma}{\Gamma, (\alpha \rightarrow \beta)^{\bullet}, \Delta, \Sigma \Rightarrow \gamma}(\bullet \rightarrow \text{left}).$$

By the hypothesis of induction, we have (1) $\dot{\mathbf{1}} \leq v(\Delta^{*}\!\rightarrow\!\alpha^{\bullet})$ and (2) $\dot{\mathbf{1}} \leq v(\Gamma^{*} * \beta^{\bullet} * \Sigma^{*}\!\rightarrow\!\gamma)$. We show $\dot{\mathbf{1}} \leq v(\Gamma^{*} * (\alpha\!\rightarrow\!\beta)^{\bullet} * \Delta^{*} * \Sigma^{*}\!\rightarrow\!\gamma)$. By (1) and (2), we obtain (3) $v(\Delta^{*}) \leq v(\alpha^{\bullet})$ and (4) $v(\beta^{\bullet}) \leq v(\Gamma^{*})\dot{\leftarrow}(v(\Sigma^{*})\dot{\rightarrow}v(\gamma))$, because

$$\begin{aligned} \dot{\mathbf{1}} \leq v(\Gamma^{*} * \beta^{\bullet} * \Sigma\!\rightarrow\!\gamma) &\quad \text{iff} \\ v(\Gamma^{*}) \cdot v(\beta^{\bullet}) \cdot v(\Sigma^{*}) \leq v(\gamma) &\quad \text{iff} \\ v(\Gamma^{*}) \cdot v(\beta^{\bullet}) \leq v(\Sigma^{*})\dot{\rightarrow}v(\gamma) &\quad \text{iff} \\ v(\beta^{\bullet}) \leq v(\Gamma^{*})\dot{\leftarrow}(v(\Sigma^{*})\dot{\rightarrow}v(\gamma)). & \end{aligned}$$

By (3), (4) and the monotonicity condition, we obtain:

$$v(\alpha^{\bullet})\dot{\rightarrow}v(\beta^{\bullet}) \leq v(\Delta^{*})\dot{\rightarrow}(v(\Gamma^{*})\dot{\leftarrow}(v(\Sigma^{*})\dot{\rightarrow}v(\gamma))).$$

We have the fact that (5) $v(\alpha^{\bullet})\dot{\rightarrow}v(\beta^{\bullet})$ iff $v(\alpha)^{\circ}\dot{\rightarrow}v(\beta)^{\circ}$ iff $(v(\alpha)\dot{\rightarrow}v(\beta))^{\circ}$ iff $v(\alpha\!\rightarrow\!\beta)^{\circ}$ iff $v((\alpha\!\rightarrow\!\beta)^{\bullet})$. We thus obtain:

$$\begin{aligned} v(\alpha^{\bullet})\dot{\rightarrow}v(\beta^{\bullet}) \leq v(\Delta^{*})\dot{\rightarrow}(v(\Gamma^{*})\dot{\leftarrow}(v(\Sigma^{*})\dot{\rightarrow}v(\gamma))) &\quad \text{iff} \\ (v(\alpha^{\bullet})\dot{\rightarrow}v(\beta^{\bullet})) \cdot v(\Delta^{*}) \leq v(\Gamma^{*})\dot{\leftarrow}(v(\Sigma^{*})\dot{\rightarrow}v(\gamma)) &\quad \text{iff} \\ v(\Gamma^{*}) \cdot (v(\alpha^{\bullet})\dot{\rightarrow}v(\beta^{\bullet})) \cdot v(\Delta^{*}) \leq v(\Sigma^{*})\dot{\rightarrow}v(\gamma) &\quad \text{iff} \\ v(\Gamma^{*}) \cdot (v(\alpha^{\bullet})\dot{\rightarrow}v(\beta^{\bullet})) \cdot v(\Delta^{*}) \cdot v(\Sigma^{*}) \leq v(\gamma) &\quad \text{iff} \\ v(\Gamma^{*}) \cdot (v(\alpha\!\rightarrow\!\beta)^{\bullet})) \cdot v(\Delta^{*}) \cdot v(\Sigma^{*}) \leq v(\gamma) \quad (\text{by}(5)) &\quad \text{iff} \\ \dot{\mathbf{1}} \leq v(\Gamma^{*} * (\alpha\!\rightarrow\!\beta)^{\bullet} * \Delta^{*} * \Sigma^{*}\!\rightarrow\!\gamma). & \end{aligned}$$

(Case ($\bullet$regu)): The last inference rule of P is of the form:

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma^{\bullet} \Rightarrow \alpha^{\bullet}}(\bullet\text{regu}).$$

[6] We remark that in this case, we have to use the operation $\dot{\leftarrow}$.

First, we show the case for $\Gamma \equiv \emptyset$, i.e. $\dot{\mathbf{1}} \leq v(\alpha)$ implies $\dot{\mathbf{1}} \leq v(\alpha^{\bullet})$. Suppose $\dot{\mathbf{1}} \leq v(\alpha)$. Then, we have $\dot{\mathbf{1}}^{\circ} \leq v(\alpha)^{\circ}$ by C2′, and hence we have $\dot{\mathbf{1}} \leq v(\alpha^{\bullet})$ by C7. Next, we show the case for $\Gamma \neq \emptyset$. In this case, we only consider the case for $\Gamma \equiv \langle \gamma_1, \gamma_2, \gamma_3 \rangle$, i.e. we show that $\dot{\mathbf{1}} \leq v(\gamma_1 * \gamma_2 * \gamma_3 \rightarrow \alpha)$ implies $\dot{\mathbf{1}} \leq v(\gamma_1^{\bullet} * \gamma_2^{\bullet} * \gamma_3^{\bullet} \rightarrow \alpha^{\bullet})$. Suppose $\dot{\mathbf{1}} \leq v(\gamma_1 * \gamma_2 * \gamma_3 \rightarrow \alpha)$. Then, we have $v(\gamma_1 * \gamma_2 * \gamma_3) \leq v(\alpha)$, and hence

$$
\begin{aligned}
v(\gamma_1 * \gamma_2 * \gamma_3) \leq v(\alpha) &\quad \text{iff} \\
v(\gamma_1 * \gamma_2 * \gamma_3)^{\circ} \leq v(\alpha)^{\circ} (\text{by C2}') &\quad \text{iff} \\
v(\gamma_1)^{\circ} \cdot v(\gamma_2)^{\circ} \cdot v(\gamma_3)^{\circ} \leq v(\alpha)^{\circ}\ (\text{by C10})^{7} &\quad \text{iff} \\
v(\gamma_1^{\bullet}) \cdot v(\gamma_2^{\bullet}) \cdot v(\gamma_3^{\bullet}) \leq v(\alpha^{\bullet}) &\quad \text{iff} \\
v(\gamma_1^{\bullet} * \gamma_2^{\bullet} * \gamma_3^{\bullet}) \leq v(\alpha^{\bullet}) &\quad \text{iff} \\
\dot{\mathbf{1}} \leq v(\gamma_1^{\bullet} * \gamma_2^{\bullet} * \gamma_3^{\bullet} \rightarrow \alpha^{\bullet}).&
\end{aligned}
$$

4.2. Proof of Theorem 3.6

We prove the completeness theorem for QIQL + (•regu) + (•regu^{-1}), which is theorem-equivalent to QIQL by Theorem 2.6, constructing a canonical twist-free-involutive-quantale model.

First, we construct a structure $\mathbf{M} := \langle M, \cdot, [], \leq \rangle$ such that

1. $M := \{[\Gamma] | [\Gamma]$ is a finite sequence of formulae$\}$,
2. $[\Gamma] \cdot [\Delta] := [\Gamma, \Delta]$ (the concatenation),
3. $[]$ is an empty sequence,
4. $[\Gamma] \leq [\Delta]$ is defined as $\vdash \Gamma \Rightarrow \Delta^{\star}$.

$[\Gamma] \doteq [\Delta]$ is defined as $[\Gamma] \leq [\Delta]$ and $[\Delta] \leq [\Gamma]$.
$\mathbf{M}$ is a *pre-ordered monoid*, i.e. the following conditions hold for $\mathbf{M}$:

1. $\langle M, \cdot, [] \rangle$ is a monoid with the identity $[]$,
2. $\langle M, \leq \rangle$ is a pre-ordered set,
3. $x_1 \leq x_2$ and $y_1 \leq y_2$ imply $x_1 \cdot y_1 \leq x_2 \cdot y_2$ for all $x_1, x_2, y_1, y_2 \in M$.

Next, we construct the power set structure $\mathbf{P(M)} := \langle P(M), \bigcup, \circ, \{[]\} \rangle$ of $\mathbf{M}$ such that

1. $P(M)$ is the power set of M,
2. $\bigcup$ is usual set theoretic infinite union (we also assume usual set theoretic operations $\cup$ and $\cap$),
3. $\circ$ is defined as
   $X \circ Y := \{x \cdot y | x \in X \quad \text{and} \quad y \in Y\} \quad \text{for all} \quad X, Y \in P(M)$.

[7] We remark that this case can not be adapted for IQL. Using the twist condition, we have $v(\gamma_1 * \gamma_2 * \gamma_3)^{\circ} = v(\gamma_3)^{\circ} \cdot v(\gamma_2)^{\circ} \cdot v(\gamma_1)^{\circ}$. Thus, we can not adopt (•regu) for IQL.

We define two operations $\dot{\rightarrow}$ and $\dot{\leftarrow}$ as

$$Y \dot{\rightarrow} Z := \{x \mid \forall y \in Y(x \cdot y \in Z)\},$$

$$Y \dot{\leftarrow} Z := \{x \mid \forall y \in Y(y \cdot x \in Z)\}$$

for all $Y, Z \in P(M)$. We assume M (the greatest element) and $\emptyset$ (the least element) as the constants in $P(M)$. We can derive the following conditions:

$$X \subseteq Y \dot{\rightarrow} Z \text{ iff } X \circ Y \subseteq Z \text{ for all } X, Y, Z \in P(M),$$

$$X \subseteq Y \dot{\leftarrow} Z \text{ iff } Y \circ X \subseteq Z \text{ for all } X, Y, Z \in P(M).$$

We then have the following.

Proposition 4.1. **P(M)** *is a quantale.*

A unary operation C on the power set $P(M)$ of M is called a *closure operation* if the following properties hold: for all $X, Y \in P(M)$,

$$X \subseteq CX,$$

$$CCX \subseteq CX,$$

$$CX \circ CY \subseteq C(X \circ Y),$$

$$X \subseteq Y \text{ implies } CX \subseteq CY.$$

X is called a *C-closed element* of $P(M)$ if $CX = X \in P(M)$.

Then, we construct a structure $\mathbf{C(P(M))} := \langle C(P(M)), \bigcup_c, \circ_c, C\{[]\}\rangle$ such that

1. C is a closure operation on $P(M)$ called a *MacNeille closure* such that $CX := (X^{\rightarrow})^{\leftarrow}$ where $X^{\rightarrow} := \{y \mid \forall x \in X(x \leq y)\}$ and $X^{\leftarrow} := \{y \mid \forall x \in X(y \leq x)\}$,
2. $C(P(M))$ is the set of all C-closed elements of $P(M)$,
3. $\bigcup_c$ is defined as $\bigcup_c X_i := C(\bigcup X_i)$ for all $X_i \in P(M)$,
4. $\circ_c$ is defined as $X \circ_c Y := C(X \circ Y)$ for all $X, Y \in P(M)$.

We assume the elements M (the greatest element) and $C\emptyset$ (the least element) of $C(P(M))$.

We remark that $C(P(M))$ is closed under the operations $\cap, \bigcup_c, \circ_c, \dot{\rightarrow}$ and $\dot{\leftarrow}$. This closure operation C has the following properties: for all $X, Y \in P(M)$,

$$C(CX \cup CY) = C(X \cup Y),$$

$$C(CX \circ CY) = C(X \circ Y),$$

$$CCX = CX.$$

Then we can show the following.

Proposition 4.2. **C(P(M))** *is a quantale.*

We can show the following.

Lemma 4.3. *Let C be the MacNeille closure on $P(M)$. Then, for any* $[\Gamma], [\Delta] \in M$,

1. $C\{[\Gamma]\} = \{[\Delta] | \vdash \Delta \Rightarrow \Gamma^\star\}$,
2. $C\{[\Gamma]\} \subseteq C\{[\Delta]\}$ iff $\vdash \Gamma \Rightarrow \Delta^\star$,
3. $C\{[(\Gamma^\star) \vee (\Delta^\star)]\} = C\{[\Gamma], [\Delta]\}$.

Proof:

(1) First, we show $C\{[\Gamma]\} \subseteq \{[\Delta] | \vdash \Delta \Rightarrow \Gamma^\star\}$. Suppose $[\Sigma] \in C\{[\Gamma]\}$. Then,

$$[\Sigma] \in (\{[\Gamma]\}^{\rightarrow})^{\leftarrow} \text{ iff}$$
$$\forall [\Pi] \in \{[\Gamma]\}^{\rightarrow} (\vdash \Sigma \Rightarrow \Pi^\star) \text{ iff}$$
$$\forall [\Pi] (\forall [\Lambda] \in \{[\Gamma]\} (\vdash \Lambda \Rightarrow \Pi^\star) \text{ implies } \vdash \Sigma \Rightarrow \Pi^\star) \text{ iff}$$
$$\forall [\Pi] (\vdash \Gamma \Rightarrow \Pi^\star \text{ implies } \vdash \Sigma \Rightarrow \Pi^\star).$$

Taking Γ for Π, we obtain $\vdash \Sigma \Rightarrow \Gamma^\star$. This means $[\Sigma] \in \{[\Delta] | \vdash \Delta \Rightarrow \Gamma^\star\}$. The converse is obvious using (cut) and ($*$left).

(2) First, we show that $C\{[\Gamma]\} \subseteq C\{[\Delta]\}$ implies $\vdash \Gamma \Rightarrow \Delta^\star$. Suppose $C\{[\Gamma]\} \subseteq C\{[\Delta]\}$ holds. Then we have $[\Gamma] \in \{[\Gamma'] | \vdash \Gamma' \Rightarrow \Gamma^\star\} \subseteq \{[\Delta'] | \vdash \Delta' \Rightarrow \Delta^\star\}$ by Lemma 4.3 (1). Therefore $\vdash \Gamma \Rightarrow \Delta^\star$. Next we show the converse. We show that, for any $[\Pi]$, if $\vdash \Pi \Rightarrow \Gamma^\star$ then $\vdash \Pi \Rightarrow \Delta^\star$. Suppose $\vdash \Pi \Rightarrow \Gamma^\star$ and $\vdash \Gamma \Rightarrow \Delta^\star$. Then we obtain $\vdash \Pi \Rightarrow \Delta^\star$ by ($*$left) and (cut).

(3) First, we show $C\{[(\Gamma^\star) \vee (\Delta^\star)]\} \subseteq C\{[\Gamma], [\Delta]\}$. Suppose $[\Sigma] \in C\{[(\Gamma^\star) \vee (\Delta^\star)]\}$. Then (*): $\vdash \Sigma \Rightarrow (\Gamma^\star) \vee (\Delta^\star)$ by Lemma 4.3 (1). We show $[\Sigma] \in C\{[\Gamma], [\Delta]\}$, that is, if $\vdash \Gamma \Rightarrow \Pi^\star$ and $\vdash \Delta \Rightarrow \Pi^\star$ then $\vdash \Sigma \Rightarrow \Pi^\star$ for any $[\Pi] \in M$ because we have that

$$[\Sigma] \in (\{[\Gamma], [\Delta]\}^{\rightarrow})^{\leftarrow} \quad \text{iff}$$
$$\forall [\Pi] \in \{[\Gamma], [\Delta]\}^{\rightarrow} (\vdash \Sigma \Rightarrow \Pi^\star) \quad \text{iff}$$
$$\forall [\Pi] (\forall [\Lambda] \in \{[\Gamma], [\Delta]\} (\vdash \Lambda \Rightarrow \Pi^\star) \text{ implies } \vdash \Sigma \Rightarrow \Pi^\star).$$

Suppose $\vdash \Gamma \Rightarrow \Pi^\star$, $\vdash \Delta \Rightarrow \Pi^\star$ and (*). We obtain $\vdash \Sigma \Rightarrow \Pi^\star$ by ($\vee$left), ($*$left) and (cut). Next we show $C\{[\Gamma], [\Delta]\} \subseteq C\{[(\Gamma^\star) \vee$

$(\Delta^\star)]\}$. Suppose $[\Sigma] \in C\{[\Gamma], [\Delta]\}$, that is, for any $[\Pi] \in M$, if $\vdash \Gamma \Rightarrow \Pi^\star$ and $\vdash \Delta \Rightarrow \Pi^\star$ then $\vdash \Sigma \Rightarrow \Pi^\star$. Taking $(\Gamma^\star) \vee (\Delta^\star)$ for $\Pi^\star$, we obtain $\vdash \Sigma \Rightarrow (\Gamma^\star) \vee (\Delta^\star)$. Therefore, $[\Sigma] \in C\{[(\Gamma^\star) \vee (\Delta^\star)]\}$ by Lemma 4.3 (1). □

We introduce a structure $\mathbf{M}^\star := \langle M, \cdot, [], \leq, \cdot^\circ\rangle$ (called a *pre-ordered monoid with twist-free- or quasi-involution*) such that

1. $\langle M, \cdot, [], \leq\rangle$ is **M**, the pre-ordered monoid,
2. $\cdot^\circ$ is a unary operation on M such that

$$[\Gamma]^\circ := [\Gamma^\bullet] = \langle \gamma^\bullet | \gamma \in \Gamma\rangle.$$

We construct the powerset structure $\mathbf{P}(\mathbf{M}^\star) := \langle P(M), \bigcup, \circ, \{[]\}, \cdot^{\circ_p}\rangle$ such that

1. $\langle P(M), \bigcup, \circ, \{[]\}\rangle$ is $\mathbf{P}(\mathbf{M})$,
2. $\cdot^{\circ_p}$ is a unary operation such that

$$X^{\circ_p} := \{[\Gamma]^\circ | [\Gamma] \in X\} \text{ for all } X \in P(M).$$

Proposition 4.4. $\mathbf{P}(\mathbf{M}^\star)$ *is a twist-free-involutive quantale.*

Proof: We only verify the conditions C1, C2, C4–C10.

(Case C1): We show $X^{\circ_p\circ_p} = X$ for any $X \in P(M)$. We have:

$$\begin{aligned} &X^{\circ_p\circ_p} \\ &= \{[\Delta]^\circ | [\Delta] \in \{[\Gamma]^\circ | [\Gamma] \in X\}\} \\ &= \{[\Gamma^{\bullet\bullet}] | [\Gamma] \in X\} \\ &= \{[\Gamma] | [\Gamma] \in X\} (\text{by } [\Gamma^{\bullet\bullet}] \doteq [\Gamma]) \\ &= X. \end{aligned}$$

(Case C2): We only consider the binary case: $(X \cup Y)^{\circ_p} = X^{\circ_p} \cup Y^{\circ_p}$ for any $X, Y \in P(M)$. We have:

$$\begin{aligned} &(X \cup Y)^{\circ_p} \\ &= \{[\Sigma]^\circ | [\Sigma] \in \{[\Pi] | [\Pi] \in X \cup Y\}\} \\ &= \{[\Pi^\bullet] | [\Pi] \in X \text{ or } [\Pi] \in Y\} \\ &= \{[\Gamma^\bullet] | [\Gamma] \in X\} \cup \{[\Delta^\bullet] | [\Delta] \in Y\} \\ &= X^{\circ_p} \cup Y^{\circ_p}. \end{aligned}$$

(Case C4): We show $(X \cap Y)^{\circ_p} = X^{\circ_p} \cap Y^{\circ_p}$ for any $X, Y \in P(M)$. We have:

$$
\begin{aligned}
&(X \cap Y)^{\circ_p} \\
&\quad = \{[\Sigma]^\circ | [\Sigma] \in X \cap Y\} \\
&\quad = \{[\Sigma^\bullet] | [\Sigma] \in X \text{ and } [\Sigma] \in Y\} \\
&\quad = \{[\Gamma^\bullet] | [\Gamma] \in X\} \cap \{[\Delta^\bullet] | [\Delta] \in Y\} \\
&\quad = X^{\circ_p} \cap Y^{\circ_p}.
\end{aligned}
$$

(Case C5): We show $(X \dot{\rightarrow} Y)^{\circ_p} = X^{\circ_p} \dot{\rightarrow} Y^{\circ_p}$ for any $X, Y \in P(M)$. We have:

$$
\begin{aligned}
&(X \dot{\rightarrow} Y)^{\circ_p} \\
&\quad = \{[\Pi]^\circ | [\Pi] \in \{[\Delta] | \forall [\Gamma] \in X([\Delta, \Gamma] \in Y)\}\} \\
&\quad = \{[\Pi^\bullet] | \forall [\Gamma] \in X([\Pi, \Gamma] \in Y)\}.
\end{aligned}
$$

On the other hand, we have:

$$
\begin{aligned}
&X^{\circ_p} \dot{\rightarrow} Y^{\circ_p} \\
&\quad = \{[\Pi'] | \forall [\Gamma'] \in X^{\circ_p}([\Pi', \Gamma'] \in Y^{\circ_p})\} \\
&\quad = \{[\Pi'] | \forall [\Gamma']([\Gamma'] = [\Gamma]^\circ \text{ and } [\Gamma] \in X)([\Pi', \Gamma'] = [\Delta]^\circ \text{ and } [\Delta] \in Y)\}.
\end{aligned}
$$

Then we take $\Gamma' \equiv \Gamma^\bullet$ and $\Pi' \equiv \Pi^\bullet$, and hence $X^{\circ_p} \dot{\rightarrow} Y^{\circ_p} = \{[\Pi^\bullet] | \forall [\Gamma] \in X([\Pi, \Gamma] \in Y)\}$.

(Case C6): We show $(X \dot{\leftarrow} Y)^{\circ_p} = X^{\circ_p} \dot{\leftarrow} Y^{\circ_p}$ for any $X, Y \in P(M)$. We have:

$$
\begin{aligned}
&(X \dot{\leftarrow} Y)^{\circ_p} \\
&\quad = \{[\Pi]^\circ | [\Pi] \in \{[\Delta] | \forall [\Gamma] \in X([\Gamma, \Delta] \in Y)\}\} \\
&\quad = \{[\Pi^\bullet] | \forall [\Gamma] \in X([\Gamma, \Pi] \in Y)\}.
\end{aligned}
$$

On the other hand, we have:

$$
\begin{aligned}
&X^{\circ_p} \dot{\leftarrow} Y^{\circ_p} \\
&\quad = \{[\Pi'] | \forall [\Gamma'] \in X^{\circ_p}([\Gamma', \Pi'] \in Y^{\circ_p})\} \\
&\quad = \{[\Pi'] | \forall [\Gamma']([\Gamma'] = [\Gamma]^\circ \text{ and } [\Gamma] \in X)([\Gamma', \Pi'] = [\Delta]^\circ \text{ and } [\Delta] \in Y)\}.
\end{aligned}
$$

Then we take $\Gamma' \equiv \Gamma^\bullet$ and $\Pi' \equiv \Pi^\bullet$, and hence $X^{\circ_p} \dot{\leftarrow} Y^{\circ_p} = \{[\Pi^\bullet] | \forall [\Gamma] \in X([\Gamma, \Pi] \in Y)\}$.

(Case C7): We have:

$$\{[]\}^{\circ_p} = \{[\Gamma]^\circ | [\Gamma] \in \{[]\}\} = \{[\Gamma]^\circ | [\Gamma] = []\} = \{[]^\circ\} = \{[]\}.$$

(Case C8): We show $M^{\circ_p} = M$.

$$M^{\circ_p}$$

$$
\begin{aligned}
&= \{[\Gamma]^{\circ} | [\Gamma] \in M\} \\
&= \{[\Gamma^{\bullet}] | [\Gamma] \in M\} \\
&= \{[\Gamma^{\bullet}] | [\Gamma^{\bullet\bullet}] \in M\} \text{ (by } [\Gamma^{\bullet\bullet}] \doteq [\Gamma]) \\
&= \{[\Pi] | [\Pi^{\bullet}] \in M\}.
\end{aligned}
$$

We have that $[\Pi^{\bullet}] \in M$ iff $[\Pi] \in M$. Then $\{[\Pi] | [\Pi^{\bullet}] \in M\} = \{[\Pi] | [\Pi] \in M\} = M$.

(Case C9): We have:

$$\emptyset^{\circ_p} = \{[\Gamma]^{\circ} | [\Gamma] \in \emptyset\} = \emptyset.$$

(Case C10): We show $(X \circ Y)^{\circ_p} = X^{\circ_p} \circ Y^{\circ_p}$ for any $X, Y \in P(M)$. We have:

$$
\begin{aligned}
&(X \circ Y)^{\circ_p} \\
&\quad = \{[\Pi]^{\circ} | [\Pi] \in X \circ Y\} \\
&\quad = \{[\Gamma^{\bullet}, \Delta^{\bullet}] | [\Gamma] \in X \text{ and } [\Delta] \in Y\} \\
&\quad = \{[\Gamma^{\bullet}] \cdot [\Delta^{\bullet}] | [\Gamma^{\bullet}] \in X^{\circ_p} \text{ and } [\Delta^{\bullet}] \in Y^{\circ_p}\} \\
&\quad = X^{\circ_p} \circ Y^{\circ_p}.
\end{aligned}
$$

□

Next, we construct $\mathbf{C}(\mathbf{P}(\mathbf{M}^{\star})) := \langle C(P(M)), \bigcup_c, \circ_c, C\{[]\}, \cdot^{\circ_c} \rangle$ such that

1. $\langle C(P(M)), \bigcup_c, \circ_c, C\{[]\} \rangle$ is $\mathbf{C}(\mathbf{P}(\mathbf{M}))$,
2. $\cdot^{\circ_c}$ is a unary operation such that

$$X^{\circ_c} := C(X^{\circ_p}) \text{ for all } X \in P(M).$$

Lemma 4.5. *Let C be the MacNeille closure on $P(M)$. Then, $(C\{[\Gamma]\})^{\circ_c} = C\{[\Gamma^{\bullet}]\}$ for any $[\Gamma] \in M$.*

Proof:

$$
\begin{aligned}
&(C\{[\Gamma]\})^{\circ_c} \\
&\quad = C((C\{[\Gamma]\})^{\circ_p}) \\
&\quad = C(\{[\Delta] | \vdash \Delta \Rightarrow \Gamma^{\star}\}^{\circ_p}) \text{ (by Lemma 4.3(1))} \\
&\quad = C\{[\Pi]^{\circ} | [\Pi] \in \{[\Delta] | \vdash \Delta \Rightarrow \Gamma^{\star}\}\} \\
&\quad = C\{[\Delta^{\bullet}] | \vdash \Delta \Rightarrow \Gamma^{\star}\} \\
&\quad = C\{[\Delta^{\bullet}] | \vdash \Delta^{\bullet} \Rightarrow \Gamma^{\star\bullet}\} \text{ (by } (\bullet\text{regu}) \text{ and } (\bullet\text{regu}^{-1})) \\
&\quad = C\{[\Delta^{\bullet}] | \vdash \Delta^{\bullet} \Rightarrow \Gamma^{\bullet\star}\} \text{ (by } [\Gamma^{\bullet\star}] \doteq [\Gamma^{\star\bullet}] \text{ and (cut))}
\end{aligned}
$$

$$= C\{[\Sigma] \mid \vdash \Sigma \Rightarrow \Gamma^{\bullet\star}\}$$
$$= C(C\{[\Gamma^{\bullet}]\}) \text{ (by Lemma 4.3(1))}$$
$$= C\{[\Gamma^{\bullet}]\}. \qquad \square$$

We may not prove the same lemma for IQL or BIQL, as presented earlier, because we must use (•regu) and (•regu^{-1}), and the fact $[\Gamma^{\bullet\star}] \doteq [\Gamma^{\star\bullet}]$, which is not compatible to the twist condition.

By using Lemmas 4.3 (2) and 4.5, we can show the following monotonicity condition for $\cdot^{\circ_c}$:

$$X \subseteq Y \text{ iff } X^{\circ_c} \subseteq Y^{\circ_c} \text{ for any } X, Y \in C(P(M)).$$

To show this condition, it is sufficient to prove the following:

$$C\{[\Gamma]\} \subseteq C\{[\Delta]\} \quad \text{iff} \quad (C\{[\Gamma]\})^{\circ_c} \subseteq (C\{[\Delta]\})^{\circ_c},$$

because we have Lemma 4.3 (3). We show this as follows.

$$C\{[\Gamma]\} \subseteq C\{[\Delta]\} \quad \text{iff}$$
$$\vdash \Gamma \Rightarrow \Delta^{\star} \text{ (by Lemma 4.3 (2))} \quad \text{iff}$$
$$\vdash \Gamma^{\bullet} \Rightarrow \Delta^{\star\bullet} \text{ (by (}\bullet\text{ regu) and (}\bullet\text{ regu}^{-1}\text{))} \quad \text{iff}$$
$$\vdash \Gamma^{\bullet} \Rightarrow \Delta^{\bullet\star} \text{ (by } [(\Delta^{\star})^{\bullet}] \doteq [(\Delta^{\bullet})^{\star}] \text{ and (cut))} \quad \text{iff}$$
$$(C\{[\Gamma]\})^{\circ_c} \subseteq (C\{[\Delta]\})^{\circ_c} \text{ (by Lemmas 4.3 (2) and 4.5).}$$

We show that $C(P(M))$ is closed under the operation $\cdot^{\circ_c}$. Suppose $X \in C(P(M))$, i.e. $X = CX$. Then by the monotonicity condition for $\cdot^{\circ_c}$, we have:

$$X^{\circ_c} = (CX)^{\circ_c} = C((CX)^{\circ_p}) = C(X^{\circ_p}) = C(C(X^{\circ_p})) = C(X^{\circ_c}).$$

Therefore, $X^{\circ_c} \in C(P(M))$.

We then have the following.

Proposition 4.6. **C(P(M*))** *is a twist-free-involutive quantale.*

Proof: We only verify the conditions C1, C2, C4–C10. It is sufficient to consider that all the elements of $C(P(M))$ are of the form $C\{[\Gamma]\}$ (i.e. $\{[\Delta] \mid \vdash \Delta \Rightarrow \Gamma^{\star}\}$), because we have the fact $C\{[\Pi_1], [\Pi_2], \ldots, [\Pi_n]\} = C\{[(\Pi_1^{\star}) \vee (\Pi_2^{\star}) \vee \cdots \vee (\Pi_n^{\star})]\}$ by Lemma 4.3 (3).

(Case C1): By Lemma 4.5 and the fact $[\Gamma^{\bullet\bullet}] \doteq [\Gamma] \in M$, we have

$$(C\{[\Gamma]\})^{\circ_c\circ_c} = C\{[\Gamma^{\bullet\bullet}]\} = C\{[\Gamma]\}.$$

(Case C2): We show $(C\{[\Gamma]\} \cup_c C\{[\Delta]\})^{\circ_c} = (C\{[\Gamma]\})^{\circ_c} \cup_c (C\{[\Delta]\})^{\circ_c}$. We can verify $[((\Gamma^{\star}) \vee (\Delta^{\star}))^{\bullet}] \doteq [(\Gamma^{\star})^{\bullet} \vee (\Delta^{\star})^{\bullet}] \doteq [(\Gamma^{\bullet})^{\star} \vee (\Delta^{\bullet})^{\star}]$ for any

$[\Gamma], [\Delta] \in M$. Then we have:

$$
\begin{aligned}
&(C\{[\Gamma]\} \cup_c C\{[\Delta]\})^{\circ_c} \\
&\qquad = (C(C\{[\Gamma]\} \cup C\{[\Delta]\}))^{\circ_c} \\
&\qquad = (C(\{[\Gamma]\} \cup \{[\Delta]\}))^{\circ_c} \\
&\qquad = (C\{[\Gamma], [\Delta]\})^{\circ_c} \\
&\qquad = (C\{[(\Gamma^\star) \vee (\Delta^\star)]\})^{\circ_c} \text{ (by Lemma 4.3(3))} \\
&\qquad = C\{[((\Gamma^\star) \vee (\Delta^\star))^\bullet]\} \text{ (by Lemma 4.5)} \\
&\qquad = C\{[(\Gamma^\star)^\bullet \vee (\Delta^\star)^\bullet]\} \\
&\qquad = C\{[(\Gamma^\bullet)^\star \vee (\Delta^\bullet)^\star]\} \\
&\qquad = C(\{[\Gamma^\bullet]\} \cup \{[\Delta^\bullet]\}) \text{ (by Lemma 4.3 (3))} \\
&\qquad = C(C\{[\Gamma^\bullet]\} \cup C\{[\Delta^\bullet]\}) \\
&\qquad = C\{[\Gamma^\bullet]\} \cup_c C\{[\Delta^\bullet]\} \\
&\qquad = (C\{[\Gamma]\})^{\circ_c} \cup_c (C\{[\Delta]\})^{\circ_c} \text{ (by Lemma 4.5).}
\end{aligned}
$$

(Case C4): We show $(C\{[\Gamma]\} \cap C\{[\Delta]\})^{\circ_c} = (C\{[\Gamma]\})^{\circ_c} \cap (C\{[\Delta]\})^{\circ_c}$. Before the proof, we show (*): $C\{[\Gamma]\} \cap C\{[\Delta]\} = C\{[(\Gamma^\star) \wedge (\Delta^\star)]\}$. Suppose $[\Pi] \in C\{[\Gamma]\} \cap C\{[\Delta]\}$. Then we have $\vdash \Pi \Rightarrow \Gamma^\star$ and $\vdash \Pi \Rightarrow \Delta^\star$ by Lemma 4.3 (1). Thus, we have $\vdash \Pi \Rightarrow (\Gamma^\star) \wedge (\Delta^\star)$ by ($\wedge$right), and hence $[\Pi] \in C\{[(\Gamma^\star) \wedge (\Delta^\star)]\}$ by Lemma 4.3 (1). We can show the converse by using (cut) and the fact that $\vdash (\Gamma^\star) \wedge (\Delta^\star) \Rightarrow \Gamma^\star$ and $\vdash (\Gamma^\star) \wedge (\Delta^\star) \Rightarrow \Delta^\star$. We can verify the fact that $[((\Gamma^\star) \wedge (\Delta^\star))^\bullet] \doteq [(\Gamma^\star)^\bullet \wedge (\Delta^\star)^\bullet] \doteq [(\Gamma^\bullet)^\star \wedge (\Delta^\bullet)^\star]$. Next we show the following required fact by using (*):

$$
\begin{aligned}
&(C\{[\Gamma]\} \cap C\{[\Delta]\})^{\circ_c} \\
&\qquad = (C\{[(\Gamma^\star) \wedge (\Delta^\star)]\})^{\circ_c} \text{ (by } (*)) \\
&\qquad = C\{[((\Gamma^\star) \wedge (\Delta^\star))^\bullet]\} \text{ (by Lemma 4.5)} \\
&\qquad = C\{[(\Gamma^\star)^\bullet \wedge (\Delta^\star)^\bullet]\} \\
&\qquad = C\{[(\Gamma^\bullet)^\star \wedge (\Delta^\bullet)^\star]\} \\
&\qquad = C\{[\Gamma^\bullet]\} \cap C\{[\Delta^\bullet]\} \text{ (by } (*)) \\
&\qquad = (C\{[\Gamma]\})^{\circ_c} \cap (C\{[\Delta]\})^{\circ_c} \text{ (by Lemma 4.5).}
\end{aligned}
$$

(Case C5): We show $(C\{[\Gamma]\} \dot{\rightarrow} C\{[\Delta]\})^{\circ_c} = (C\{[\Gamma]\})^{\circ_c} \dot{\rightarrow} (C\{[\Delta]\})^{\circ_c}$. Before the proof, we show (*): $C\{[(\Gamma^\star) \rightarrow (\Delta^\star)]\} = C\{[\Gamma]\} \dot{\rightarrow} C\{[\Delta]\}$. Suppose $[\Lambda] \in C\{[(\Gamma^\star) \rightarrow (\Delta^\star)]\}$, that is, $\vdash \Lambda \Rightarrow (\Gamma^\star) \rightarrow (\Delta^\star)$, and hence, $\vdash \Lambda, \Gamma^\star \Rightarrow \Delta^\star$.

We will show $[\Lambda] \in C\{[\Gamma]\}\dot{\rightarrow}C\{[\Delta]\}$, that is, $[\Lambda] \in \{\Psi_1 | \vdash \Psi_1 \Rightarrow \Gamma^\star\}\dot{\rightarrow}\{\Psi_2 | \vdash \Psi_2 \Rightarrow \Delta^\star\}$ by Lemma 4.3 (1), and hence (**): $\forall[\Pi](\vdash \Pi \Rightarrow \Gamma^\star$ implies $\vdash \Lambda, \Pi \Rightarrow \Delta^\star)$. By $\vdash \Lambda, \Gamma^\star \Rightarrow \Delta^\star$, $\vdash \Pi \Rightarrow \Gamma^\star$ and (cut), we obtain $\vdash \Lambda, \Pi \Rightarrow \Delta^\star$. Next we show the converse. Suppose $[\Lambda] \in C\{[\Gamma]\}\dot{\rightarrow}C\{[\Delta]\}$, that is, (**). We will show $[\Lambda] \in C\{[(\Gamma^\star)\rightarrow(\Delta^\star)]\}$, that is, $\vdash \Lambda, \Gamma^\star \Rightarrow \Delta^\star$. Taking $\Gamma^\star$ for Π in (**), we have $\vdash \Lambda, \Gamma^\star \Rightarrow \Delta^\star$. Moreover, we can verify the fact $[((\Gamma^\star)\rightarrow(\Delta^\star))^\bullet] \doteq [(\Gamma^\star)^\bullet\rightarrow(\Delta^\star)^\bullet] \doteq [(\Gamma^\bullet)^\star\rightarrow(\Delta^\bullet)^\star] \in M$. Next we show the required fact by using (*):

$$
\begin{aligned}
&(C\{[\Gamma]\}\dot{\rightarrow}C\{[\Delta]\})^{\circ_c}\\
&\qquad = (C\{[(\Gamma^\star)\rightarrow(\Delta^\star)]\})^{\circ_c} \text{ (by } (*))\\
&\qquad = C\{[((\Gamma^\star)\rightarrow(\Delta^\star))^\bullet]\} \text{ (by Lemma 4.5)}\\
&\qquad = C\{[(\Gamma^\star)^\bullet\rightarrow(\Delta^\star)^\bullet]\}\\
&\qquad = C\{[(\Gamma^\bullet)^\star\rightarrow(\Delta^\bullet)^\star]\}\\
&\qquad = C\{[\Gamma^\bullet]\}\dot{\rightarrow}C\{[\Delta^\bullet]\} \text{ (by } (*))\\
&\qquad = (C\{[\Gamma]\})^{\circ_c}\dot{\rightarrow}(C\{[\Delta]\})^{\circ_c} \text{ (by Lemma 4.5).}
\end{aligned}
$$

(Case C6): We show $(C\{[\Gamma]\}\dot{\leftarrow}C\{[\Delta]\})^{\circ_c} = (C\{[\Gamma]\})^{\circ_c}\dot{\leftarrow}(C\{[\Delta]\})^{\circ_c}$. Before the proof, we show (*): $C\{[(\Gamma^\star)\leftarrow(\Delta^\star)]\} = C\{[\Gamma]\}\dot{\leftarrow}C\{[\Delta]\}$. Suppose $[\Lambda] \in C\{[(\Gamma^\star)\leftarrow(\Delta^\star)]\}$, that is, $\vdash \Lambda \Rightarrow (\Gamma^\star)\leftarrow(\Delta^\star)$, and hence, $\vdash \Gamma^\star, \Lambda \Rightarrow \Delta^\star$. We will show $[\Lambda] \in C\{[\Gamma]\}\dot{\leftarrow}C\{[\Delta]\}$, that is, $[\Lambda] \in \{\Psi_1 | \vdash \Psi_1 \Rightarrow \Gamma^\star\}\dot{\leftarrow}\{\Psi_2 | \vdash \Psi_2 \Rightarrow \Delta^\star\}$ by Lemma 4.3 (1), and hence (**): $\forall[\Pi](\vdash \Pi \Rightarrow \Gamma^\star$ implies $\vdash \Pi, \Lambda \Rightarrow \Delta^\star)$. By $\vdash \Gamma^\star, \Lambda \Rightarrow \Delta^\star$, $\vdash \Pi \Rightarrow \Gamma^\star$ and (cut), we obtain $\vdash \Pi, \Lambda \Rightarrow \Delta^\star$. Next we show the converse. Suppose $[\Lambda] \in C\{[\Gamma]\}\dot{\leftarrow}C\{[\Delta]\}$, that is, (**). We will show $[\Lambda] \in C\{[(\Gamma^\star)\leftarrow(\Delta^\star)]\}$, that is, $\vdash \Gamma^\star, \Lambda \Rightarrow \Delta^\star$. Taking $\Gamma^\star$ for Π in (**), we have $\vdash \Gamma^\star, \Lambda \Rightarrow \Delta^\star$. Moreover, we can verify the fact $[((\Gamma^\star)\leftarrow(\Delta^\star))^\bullet] \doteq [(\Gamma^\star)^\bullet\leftarrow(\Delta^\star)^\bullet] \doteq [(\Gamma^\bullet)^\star\leftarrow(\Delta^\bullet)^\star] \in M$. Next we show the required fact by using (*):

$$
\begin{aligned}
&(C\{[\Gamma]\}\dot{\leftarrow}C\{[\Delta]\})^{\circ_c}\\
&\qquad = (C\{[(\Gamma^\star)\leftarrow(\Delta^\star)]\})^{\circ_c} \text{ (by } (*))\\
&\qquad = C\{[((\Gamma^\star)\leftarrow(\Delta^\star))^\bullet]\} \text{ (by Lemma 4.5)}\\
&\qquad = C\{[(\Gamma^\star)^\bullet\leftarrow(\Delta^\star)^\bullet]\}\\
&\qquad = C\{[(\Gamma^\bullet)^\star\leftarrow(\Delta^\bullet)^\star]\}\\
&\qquad = C\{[\Gamma^\bullet]\}\dot{\leftarrow}C\{[\Delta^\bullet]\} \text{ (by } (*))\\
&\qquad = (C\{[\Gamma]\})^{\circ_c}\dot{\leftarrow}(C\{[\Delta]\})^{\circ_c} \text{ (by Lemma 4.5).}
\end{aligned}
$$

(Case C7): $(C\{[]\})^{\circ_c} = C\{[]^\circ\} = C\{[]\}$.

(Case C8): $M^{\circ_c} = C(M^{\circ_p}) = CM = M$.
(Case C9): We have:

$$
\begin{aligned}
&(C\emptyset)^{\circ_c} \\
&\quad = C((C\emptyset)^{\circ_p}) \\
&\quad = C((C\{[\bot]\})^{\circ_p}) \\
&\quad = C(\{[\Delta] | \vdash \Delta \Rightarrow \bot\}^{\circ_p}) \text{ (by Lemma 4.3 (1))} \\
&\quad = C\{[\Delta]^{\circ} | \vdash \Delta \Rightarrow \bot\} \\
&\quad = C\{[\Delta^{\bullet}] | \vdash \Delta^{\bullet} \Rightarrow \bot^{\bullet}\} \text{ (by ($\bullet$regu) and ($\bullet$regu}^{-1}\text{))} \\
&\quad = C\{[\Pi] | \vdash \Pi \Rightarrow \bot^{\bullet}\} \\
&\quad = C(C\{[\bot^{\bullet}]\}) \text{ (by Lemma 4.3 (1))} \\
&\quad = C\{[\bot^{\bullet}]\} \\
&\quad = C\{[\bot]\} \\
&\quad = C\emptyset.
\end{aligned}
$$

(Case C10): We show $(C\{[\Gamma]\} \circ_c C\{[\Delta]\})^{\circ_c} = (C\{[\Gamma]\})^{\circ_c} \circ_c (C\{[\Delta]\})^{\circ_c})$. We have:

$$
\begin{aligned}
&(C\{[\Gamma]\} \circ_c C\{[\Delta]\})^{\circ_c} \\
&\quad = (C(C\{[\Gamma]\} \circ C\{[\Delta]\}))^{\circ_c} \\
&\quad = (C(\{[\Gamma]\} \circ \{[\Delta]\}))^{\circ_c} \\
&\quad = (C\{[\Gamma, \Delta]\})^{\circ_c} \\
&\quad = C\{[\Gamma^{\bullet}, \Delta^{\bullet}]\} \text{ (by Lemma 4.5)} \\
&\quad = C(\{[\Gamma^{\bullet}]\} \circ \{[\Delta^{\bullet}]\}) \\
&\quad = C(C\{[\Gamma^{\bullet}]\} \circ C\{[\Delta^{\bullet}]\}) \\
&\quad = C\{[\Gamma^{\bullet}]\} \circ_c C\{[\Delta^{\bullet}]\} \\
&\quad = (C\{[\Gamma]\})^{\circ_c} \circ_c (C\{[\Delta]\})^{\circ_c} \text{ (by Lemma 4.5).}
\end{aligned}
$$

□

Next we define a *valuation* on **C**(**P**(**M***)). A valuation v on **C**(**P**(**M***)) is a mapping from the set of all propositional variables to $C(P(M))$ such that

$$v(p) := C\{[p]\}.$$

We can extend to the mapping from the set Φ of all formulae to $C(P(M))$, that is, we can prove the following by induction on the complexity of $\alpha \in \Phi$:

$$v(\alpha) = C\{[\alpha]\}.$$

Here, we only show the case $\alpha \equiv \beta^{\bullet}$, i.e. we show $v(\beta^{\bullet}) = C\{[\beta^{\bullet}]\}$. This case is proved using the induction hypothesis and Lemma 4.5 as follows:

$$v(\beta^{\bullet}) = v(\beta)^{\circ_c} = (C\{[\beta]\})^{\circ_c} = C\{[\beta^{\bullet}]\}.$$

This completes the construction of a canonical twist-free-involutive-quantale model for QIQL. Using this model, we can prove the required completeness theorem for QIQL.

5. PHASE MODELS

Definition 5.1. (Intuitionistic non-commutative phase space) An *intuitionistic non-commutative phase space* is a structure $\langle \mathbf{M}, \mathrm{cl} \rangle$ satisfying the following conditions:

1. $\mathbf{M} := \langle M, \cdot, 1 \rangle$ is a monoid with the identity 1,
2. cl is a closure operation on $P(M)$ such that, for any $X, Y \in P(M)$,
   C1: $X \subseteq \mathrm{cl}(X)$,
   C2: $\mathrm{clcl}(X) \subseteq \mathrm{cl}(X)$,
   C3: $X \subseteq Y$ implies $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$,
   C4: $\mathrm{cl}(X) \circ \mathrm{cl}(Y) \subseteq \mathrm{cl}(X \circ Y)$,
   where the operation $\circ$ is defined as $X \circ Y := \{x \cdot y | x \in X \text{ and } y \in Y\}$.

Definition 5.2. (Intuitionistic non-commutative phase structure) We define constants and operations on $P(M)$ as follows: for any $X, Y \in P(M)$,

$$\dot{\mathbf{1}} := \mathrm{cl}\{1\},$$

$$\dot{\top} := M,$$

$$\dot{\bot} := \mathrm{cl}(\emptyset),$$

$$X \dot{\rightarrow} Y := \{y | \forall x \in X (y \cdot x \in Y)\},$$

$$X \dot{\leftarrow} Y := \{y | \forall x \in X (x \cdot y \in Y)\},$$

$$X \dot{\wedge} Y := X \cap Y,$$

$$X \dot{\vee} Y := \mathrm{cl}(X \cup Y),$$

$$X \dot{*} Y := \mathrm{cl}(X \circ Y).$$

We define $D := \{X \in P(M) | X = \mathrm{cl}(X)\}$. Then

$$\mathbf{D} := \langle D, \dot{\rightarrow}, \dot{\leftarrow}, \dot{*}, \dot{\wedge}, \dot{\vee}, \dot{\mathbf{1}}, \dot{\top}, \dot{\bot}\rangle$$

is called an *intuitionistic non-commutative phase structure*.[8]

We remark that D is closed under the operations $\dot{\rightarrow}$, $\dot{\leftarrow}$, $\dot{*}$, $\dot{\wedge}$ and $\dot{\vee}$, and $\dot{\mathbf{1}}, \dot{\top}, \dot{\bot} \in D$.

Definition 5.3. (Involutive and twist-free-involutive valuations) *Involutive valuations* v^+ and v^- on an intuitionistic non-commutative phase structure $\mathbf{D} := \langle D, \dot{\rightarrow}, \dot{\leftarrow}, \dot{*}, \dot{\wedge}, \dot{\vee}, \dot{\mathbf{1}}, \dot{\top}, \dot{\bot}\rangle$ are mappings from the set of all propositional variables to D. Then, v^+ and v^- are extended to mappings from the set of all formulae to D by

1. $v^+(\mathbf{1}) := \dot{\mathbf{1}}$,
2. $v^+(\top) := \dot{\top}$,
3. $v^+(\bot) := \dot{\bot}$,
4. $v^+(\alpha \wedge \beta) := v^+(\alpha)\dot{\wedge}v^+(\beta)$,
5. $v^+(\alpha \vee \beta) := v^+(\alpha)\dot{\vee}v^+(\beta)$,
6. $v^+(\alpha * \beta) := v^+(\alpha)\dot{*}v^+(\beta)$,
7. $v^+(\alpha\rightarrow\beta) := v^+(\alpha)\dot{\rightarrow}v^+(\beta)$,
8. $v^+(\alpha\leftarrow\beta) := v^+(\alpha)\dot{\leftarrow}v^+(\beta)$,
9. $v^+(\alpha^\bullet) := v^-(\alpha)$,
10. $v^-(\mathbf{1}) := \dot{\mathbf{1}}$,
11. $v^-(\top) := \dot{\top}$,
12. $v^-(\bot) := \dot{\bot}$,
13. $v^-(\alpha \wedge \beta) := v^-(\alpha)\dot{\wedge}v^-(\beta)$,
14. $v^-(\alpha \vee \beta) := v^-(\alpha)\dot{\vee}v^-(\beta)$,
15. $v^-(\alpha * \beta) := v^-(\beta)\dot{*}v^-(\alpha)$,
16. $v^-(\alpha\rightarrow\beta) := v^-(\alpha)\dot{\rightarrow}v^-(\beta)$,
17. $v^-(\alpha\leftarrow\beta) := v^-(\alpha)\dot{\leftarrow}v^-(\beta)$,
18. $v^-(\alpha^\bullet) := v^+(\alpha)$.

Twist-free-involutive valuations v^+ and v^- on $\mathbf{D}$ are defined in a similar way, but the negative valuation v^- is obtained from that for the involutive valuations by replacing the condition 15 by

19. $v^-(\alpha * \beta) := v^-(\alpha)\dot{*}v^-(\beta)$.

An intuitive meaning of the involutive or twist-free-involutive valuations is that, for a quantum $\{0, 1\}$-analogy, v^+ and v^- respectively correspond to *provability in the 1-state* and *provability in the 0-state*.

[8] An intuitionistic non-commutative phase structure as a model of non-commutative intuitionistic linear logic was established by Abrusci (1990). Another more general algebraic framework, called *pretopology*, was also established by Sambin (1995).

Definition 5.4. (Intuitionistic non-commutative phase model) An *intuitionistic non-commutative phase model* for BIQL (QIQL) is a structure $\langle \mathbf{D}, v^+, v^- \rangle$ such that $\mathbf{D}$ is an intuitionistic non-commutative phase structure, and v^+ and v^- are involutive valuations (twist-free-involutive valuations, respectively). A formula α is *true* in an intuitionistic non-commutative phase model $\langle \mathbf{D}, v^+, v^- \rangle$ for BIQL (QIQL) if $\dot{\mathbf{1}} \subseteq v^+(\alpha)$ (or equivalently $1 \in v^+(\alpha)$) holds, and *involutive valid* (*twist-free-involutive valid*) in an intuitionistic non-commutative phase structure $\mathbf{D}$ if it is true for any involutive valuations (twist-free-involutive valuations, respectively) v^+ and v^- on the intuitionistic non-commutative phase structure. A sequent $\alpha_1, \ldots, \alpha_n \Rightarrow \beta$ (or $\Rightarrow \beta$) is *true* in an intuitionistic non-commutative phase model $\langle \mathbf{D}, v^+, v^- \rangle$ for BIQL (QIQL) if the formula $\alpha_1 * \cdots * \alpha_n \rightarrow \beta$ (or β) is true in it, and involutive valid (twist-free-involutive valid, respectively) in an intuitionistic non-commutative phase structure if so is $\alpha_1 * \cdots * \alpha_n \rightarrow \beta$ (or β).

Theorem 5.6. (Soundness and completeness for BIQL and QIQL) *Let C be the class of all intuitionistic non-commutative phase structures, $L_1(C) := \{S \mid$ a sequent S is involutive valid in all intuitionistic non-commutative phase structures of $C\}$, $L_2(C) := \{S \mid$ a sequent S is twist-free-involutive valid in all intuitionistic non-commutative phase structures of $C\}$, $L_1 := \{S \mid$ BIQL$\vdash S\}$ and $L_2 := \{S \mid$ QIQL$\vdash S\}$. Then, $L_1 = L_1(C)$ and $L_2 = L_2(C)$.*

The proof of this main theorem will be given in the next section.

Since the difference between the phase model for FL and the proposed phase model for QIQL is only the use of the negative valuation v^-, this theorem means semantically that twist-free-involutive quantales are essentially equivalent to quantales.

6. PROOF OF THEOREM 5.5

The proof of the soundness part is straightforward, and hence is omitted. Using a modified version of the method by Okada (2002), we can show the completeness parts for BIQL and QIQL, and can obtain the cut-elimination theorems for these logics at the same time. The proof is only given for BIQL in the following.

Definition 6.1. We define a monoid $\langle M, \cdot, 1 \rangle$ as follows:

1. $M := \{[\Gamma] | [\Gamma]$ is a finite sequence of formulae$\}$,
2. $[\Gamma] \cdot [\Delta] := [\Gamma, \Delta]$,
3. $1 := []$.

We define the following: for any formula α,

$$\|\alpha\|^+ := \{[\Gamma] | \vdash_{\mathrm{cf}} \Gamma \Rightarrow \alpha\},$$

$$\|\alpha\|^- := \{[\Gamma] | \vdash_{\mathrm{cf}} \Gamma \Rightarrow \alpha^\bullet\}$$

where $\vdash_{\mathrm{cf}}$ means "cut-free provable in BIQL." We have the fact

$$\|\alpha\|^+ = \|\alpha^\bullet\|^-$$

for any formula α. This fact is verified using the rules ($\bullet$right) and ($\bullet$right^{-1}), where ($\bullet$right^{-1}) is admissible in *cut-free* BIQL. We then define

$$D := \{X | X = \bigcap_{i \in I} \|\alpha_i\|^+\} = \{X | X = \bigcap_{i \in I} \|\beta_i\|^-\}$$

for arbitrary indexing set I, and arbitrary formula α_i and $\beta_i \equiv \alpha_i^\bullet$. Then, we define

$$\mathrm{cl}(X) := \bigcap \{Y \in D | X \subseteq Y\}.$$

We define the following constants and operations on $P(M)$: for any $X, Y \in P(M)$,

$$\dot{\mathbf{1}} := \mathrm{cl}\{1\},$$

$$\dot{\top} := M,$$

$$\dot{\bot} := \mathrm{cl}(\emptyset),$$

$$X \dot{\rightarrow} Y := \{[\Delta] | \forall [\Gamma] \in X([\Delta, \Gamma] \in Y)\},$$

$$X \dot{\leftarrow} Y := \{[\Delta] | \forall [\Gamma] \in X([\Gamma, \Delta] \in Y)\},$$

$$X \dot{\wedge} Y := X \cap Y,$$

$$X \dot{\vee} Y := \mathrm{cl}(X \cup Y),$$

$$X \dot{*} Y := \mathrm{cl}(X \circ Y) \text{ where } X \circ Y := \{[\Gamma, \Delta] | [\Gamma] \in X \text{ and } [\Delta] \in Y\}.$$

Involutive valuations v^+ and v^- are mappings from the set of all propositional variables to D such that

$$v^+(p) := \|p\|^+,$$

$$v^-(p) := \|p\|^-$$

for any propositional variable p.

We have the following: for any $X, Y, Z \in P(M)$,

$$X \circ Y \subseteq Z \text{ iff } X \subseteq Y \dot{\rightarrow} Z,$$

$$Y \circ X \subseteq Z \text{ iff } X \subseteq Y \dot{\leftarrow} Z.$$

We remark that D is closed under arbitrary $\bigcap$.

Lemma 6.2. *Let D be defined earlier and $D_c := \{X \in P(M) | X = \mathrm{cl}(X)\}$. Then, $D = D_c$.*

Proof: First, we show $D_c \subseteq D$. Suppose $X \in D_c$. Then $X = \mathrm{cl}(X) = \bigcap\{Y \in D | X \subseteq Y\} \in D$. Next, we show $D \subseteq D_c$. Suppose $X \in D$. We show $X \in D_c$, i.e. $X = \bigcap\{Y \in D | X \subseteq Y\}$. To show this, it is sufficient to prove that

(1) $X \subseteq \{[\Gamma] | \forall W[W \in D \text{ and } X \subseteq W \text{ imply } [\Gamma] \in W]\}$,
(2) $\{[\Gamma] | \forall W[W \in D \text{ and } X \subseteq W \text{ imply } [\Gamma] \in W]\} \subseteq X$.

First, we show (1). Suppose $[\Delta] \in X$ and asuume $W \in D$ and $X \subseteq W$ for any W. Then we have $[\Delta] \in X \subseteq W$. Next we show (2). Suppose $[\Delta] \in \{[\Gamma] | \forall W[W \in D$ and $X \subseteq W$ imply $[\Gamma] \in W]\}$. By the assumption $X \in D$ and the fact $X \subseteq X$, we have $[\Delta] \in X$. □

Lemma 6.3. *For any $X, Y \in P(M)$, if $X \subseteq M$ and $Y \in D$, then $X \dot{\rightarrow} Y \in D$ and $X \dot{\leftarrow} Y \in D$.*

Proof: We show only $X \dot{\leftarrow} Y \in D$ using the assumptions. Before the proof, we remark that the rules

$$\frac{\Gamma \Rightarrow \alpha \leftarrow \beta}{\alpha, \Gamma \Rightarrow \beta}(\leftarrow \text{right}^{-1}) \quad \frac{\Gamma, \alpha * \beta, \Delta \Rightarrow \gamma}{\Gamma, \alpha, \beta, \Delta \Rightarrow \gamma}(*\text{left}^{-1})$$

are admissible in *cut-free* BIQL.

Suppose $X \subseteq M$ and $Y \in D$. We have:

$$\begin{aligned}
& X \dot{\leftarrow} Y \\
&= X \dot{\leftarrow} \bigcap_{i \in I} \|\alpha_i\|^+ \\
&= \{[\Delta] | \forall [\Gamma] \in X([\Gamma, \Delta] \in \{[\Pi] | \forall i \in I([\Pi] \in \|\alpha_i\|^+)\})\} \\
&= \{[\Delta] | \forall [\Gamma] \in X(\forall i \in I(\vdash_{\mathrm{cf}} \Gamma, \Delta \Rightarrow \alpha_i))\} \\
&= \{[\Delta] | \forall [\Gamma] \in X(\forall i \in I(\vdash_{\mathrm{cf}} \Delta \Rightarrow \Gamma^* \leftarrow \alpha_i))\} \\
&\quad \text{(by using } (*\text{left}), (*\text{left}^{-1}), (\leftarrow\text{right}) \text{ and } (\leftarrow\text{right}^{-1})) \\
&= \{[\Delta] | \forall [\Gamma] \in X(\forall i \in I([\Delta] \in \|\Gamma^* \leftarrow \alpha_i\|^+))\} \\
&= \bigcap\{\|\Gamma^* \leftarrow \alpha_i\|^+ | i \in I \text{ and } [\Gamma] \in X\} \in D.
\end{aligned}$$

□

Then we can show the following.

Proposition 6.4. *The structure $\mathbf{D} := \langle D, \dot{\rightarrow}, \dot{\leftarrow}, \dot{*}, \dot{\wedge}, \dot{\vee}, \dot{\mathbf{1}}, \dot{\top}, \dot{\bot} \rangle$ defined above forms an intuitionistic non-commutative phase structure.*

Proof: We can verify that D is closed under $\dot{\rightarrow}$, $\dot{\leftarrow}$, $\dot{*}$, $\dot{\wedge}$ and $\dot{\vee}$. In particular, for $\dot{\rightarrow}$ and $\dot{\leftarrow}$, we use Lemma 6.3. The fact $\dot{\mathbf{1}}, \dot{\top}, \dot{\bot} \in D$ is obvious. We can verify that the conditions C1–C4 for closure operation hold for this structure. The conditions C1–C3 are obvious. We only show C4: $\mathrm{cl}(X) \circ \mathrm{cl}(Y) \subseteq \mathrm{cl}(X \circ Y)$ for any $X, Y \in P(M)$. We assume the following facts, which will be proved later: for any $X, Y \in P(M)$,

(*) $\mathrm{cl}(X) \cdot Y \subseteq \mathrm{cl}(X \circ Y)$,
(**) $X \cdot \mathrm{cl}(Y) \subseteq \mathrm{cl}(X \circ Y)$.

By using the facts (*) and (**) and Lemma 6.2, we have:

$$\mathrm{cl}(X) \circ \mathrm{cl}(Y) \subseteq \mathrm{cl}(\mathrm{cl}(X) \circ Y) \subseteq \mathrm{cl}(\mathrm{cl}(X \circ Y)) = \mathrm{cl}(X \circ Y).$$

We show the remained facts (*) and (**). We have $X \circ Y \subseteq \mathrm{cl}(X \circ Y)$ by the condition C1, and hence $X \subseteq Y \dot{\rightarrow} \mathrm{cl}(X \circ Y)$ and $Y \subseteq X \dot{\leftarrow} \mathrm{cl}(X \circ Y)$ hold. Moreover, by the condition C3, we have $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y \dot{\rightarrow} \mathrm{cl}(X \circ Y))$ and $\mathrm{cl}(Y) \subseteq \mathrm{cl}(X \dot{\leftarrow} \mathrm{cl}(X \circ Y))$. Here, by $\mathrm{cl}(X \circ Y) \in D$ and Lemma 6.3, we have $Y \dot{\rightarrow} \mathrm{cl}(X \circ Y) \in D$ and $X \dot{\leftarrow} \mathrm{cl}(X \circ Y) \in D$. Thus, we obtain

$$\mathrm{cl}(X) \subseteq \mathrm{cl}(Y \dot{\rightarrow} \mathrm{cl}(X \circ Y)) = Y \dot{\rightarrow} \mathrm{cl}(X \circ Y),$$

$$\mathrm{cl}(Y) \subseteq \mathrm{cl}(X \dot{\leftarrow} \mathrm{cl}(X \circ Y)) = X \dot{\leftarrow} \mathrm{cl}(X \circ Y)$$

by Lemma 6.2. Therefore, we obtain the required facts. □

Lemma 6.5. (Key lemma) *Let α be any formula. Then,*

(1) $[\alpha] \in v^{+}(\alpha) \subseteq \|\alpha\|^{+}$,
(2) $[\alpha^{\bullet}] \in v^{-}(\alpha) \subseteq \|\alpha\|^{-}$.

Proof: We can prove this lemma by (simultaneous) induction on the complexcity of α. We demonstrate some cases for the induction step for (2).

(Case $\alpha \equiv \beta^{\bullet}$ for (2)): First we show $[\beta^{\bullet\bullet}] \in v^{-}(\beta^{\bullet})$. By the induction hypothesis for (1), we have

$$[\beta] \in v^{+}(\beta) = \bigcap_{i \in I} \|\delta_i\|^{+} = \{[\Delta] | \forall i \in I([\Delta] \in \|\delta_i\|^{+})\}.$$

Thus, we obtain:

$$\begin{aligned}
&\forall i \in I([\beta] \in \|\delta_i\|^{+}) \quad \text{iff} \\
&\forall i \in I(\vdash_{\mathrm{cf}} \beta \Rightarrow \delta_i) \quad \text{implies} \\
&\forall i \in I(\vdash_{\mathrm{cf}} \beta^{\bullet\bullet} \Rightarrow \delta_i)\,(\text{by } (\bullet\text{left})) \quad \text{iff} \\
&[\beta^{\bullet\bullet}] \in v^{+}(\beta) = v^{-}(\beta^{\bullet}).
\end{aligned}$$

Next, we show $v^-(\beta^\bullet) \subseteq \|\beta^\bullet\|^-$. Suppose $[\Gamma] \in v^-(\beta^\bullet)$. Then we have $[\Gamma] \in v^-(\beta^\bullet) = v^+(\beta) \subseteq \|\beta\|^+$ by the induction hypothesis for (1). This means $\vdash_{\text{cf}} \Gamma \Rightarrow \beta$, and hence we obtain $\vdash_{\text{cf}} \Gamma \Rightarrow \beta^{\bullet\bullet}$ by ($\bullet$right). Therefore, $[\Gamma] \in \|\beta^\bullet\|^-$.

(Case $\alpha \equiv \beta * \gamma$ for (2)): We show $[(\beta * \gamma)^\bullet] \in v^-(\beta * \gamma) \subseteq \|\beta * \gamma\|^-$.

First, we show $[(\beta * \gamma)^\bullet] \in v^-(\beta * \gamma)$, i.e.

$$[(\beta * \gamma)^\bullet] \in v^-(\beta * \gamma) \quad \text{iff}$$
$$[(\beta * \gamma)^\bullet] \in v^-(\gamma) \dot{*} v^-(\beta) \quad \text{iff}$$
$$[(\beta * \gamma)^\bullet] \in \text{cl}(v^-(\gamma) \circ v^-(\beta)) \quad \text{iff}$$
$$[(\beta * \gamma)^\bullet] \in \bigcap\{Y \in D | v^-(\gamma) \circ v^-(\beta) \subseteq Y\} \quad \text{iff}$$
$$\forall W[W \in D \text{ and } v^-(\gamma) \circ v^-(\beta) \subseteq W \text{ imply } [(\beta * \gamma)^\bullet] \in W].$$

Suppose $W \in D$ and $v^-(\gamma) \circ v^-(\beta) \subseteq W$. By the induction hypothesis, we have $[\beta^\bullet] \in v^-(\beta)$ and $[\gamma^\bullet] \in v^-(\gamma)$. Hence, we have

$$[\gamma^\bullet, \beta^\bullet] \in v^-(\gamma) \circ v^-(\beta) \subseteq W = \bigcap_{i \in I} \|\delta_i\|^+ \in D.$$

Thus, we obtain $[\gamma^\bullet, \beta^\bullet] \in \bigcap_{i \in I} \|\delta_i\|^+ = \{[\Delta] | \forall i \in I([\Delta] \in \|\delta_i\|^+)\}$, i.e. $\forall i \in I(\vdash_{\text{cf}} \gamma^\bullet, \beta^\bullet \Rightarrow \delta_i)$. Then, we have $\forall i \in I(\vdash_{\text{cf}} (\beta * \gamma)^\bullet \Rightarrow \delta_i)$ by ($\bullet *$left). Therefore, $[(\beta * \gamma)^\bullet] \in \bigcap_{i \in I} \|\delta_i\|^+ = W$.

Second, we show $v^-(\beta * \gamma) \subseteq \|\beta * \gamma\|^-$. Suppose $[\Gamma] \in v^-(\beta * \gamma)$. We show $[\Gamma] \in \|\beta * \gamma\|^-$. For the assumption, we have

$$[\Gamma] \in v^-(\beta * \gamma) \quad \text{iff}$$
$$[\Gamma] \in v^-(\gamma) \dot{*} v^-(\beta) \quad \text{iff}$$
$$[\Gamma] \in \text{cl}(v^-(\gamma) \circ v^-(\beta)) \quad \text{iff}$$
$$[\Gamma] \in \bigcap\{Y \in D | v^-(\gamma) \circ v^-(\beta) \subseteq Y\} \quad \text{iff}$$
$$\forall W[W \in D \text{ and } v^-(\gamma) \circ v^-(\beta) \subseteq W \text{ imply } [\Gamma] \in W].$$

For this, if $W = \|\beta * \gamma\|^-$, then $[\Gamma] \in \|\beta * \gamma\|^-$. Thus, it is sufficient to prove that $v^-(\gamma) \circ v^-(\beta) \subseteq \|\beta * \gamma\|^-$. Now we prove this. Suppose $[\Delta] \in v^-(\gamma) \circ v^-(\beta)$. Then $[\Delta] \equiv [\Delta_1, \Delta_2]$, $[\Delta_1] \in v^-(\gamma)$ and $[\Delta_2] \in v^-(\beta)$. By the induction hypothesis, we have $[\Delta_1] \in v^-(\gamma) \subseteq \|\gamma\|^-$ and $[\Delta_2] \in v^-(\beta) \subseteq \|\beta\|^-$, and hence $\vdash_{\text{cf}} \Delta_1 \Rightarrow \gamma^\bullet$ and $\vdash_{\text{cf}} \Delta_2 \Rightarrow \beta^\bullet$. By applying ($\bullet *$right) to these, we have $\vdash_{\text{cf}} \Delta \Rightarrow (\beta * \gamma)^\bullet$. This means $[\Delta] \in \|\beta * \gamma\|^-$.

(Case $\alpha \equiv \beta \vee \gamma$ for (2)): First, we show $[(\beta \vee \gamma)^\bullet] \in v^-(\beta \vee \gamma)$, i.e. $[(\beta \vee \gamma)^\bullet] \in v^-(\beta \vee \gamma) = v^-(\beta) \dot{\vee} v^-(\gamma) = \text{cl}(v^-(\beta) \cup v^-(\gamma)) = \cap\{Y \in D | v^-(\beta) \cup v^-(\gamma) \subseteq Y\}$. Thus, we show

$$\forall W[W \in D \text{ and } v^-(\beta) \cup v^-(\gamma) \subseteq W \text{ imply } [(\beta \vee \gamma)^\bullet] \in W].$$

Suppose $W \in D$ and $v^-(\beta) \cup v^-(\gamma) \subseteq W$, and the induction hypothesis $[\beta^\bullet] \in v^-(\beta)$ and $[\gamma^\bullet] \in v^-(\gamma)$. Then, we have

$$[\gamma^\bullet], [\beta^\bullet] \in v^-(\beta) \cup v^-(\gamma) \subseteq W = \bigcap_{i \in I} \|\delta_i\|^+ = \{[\Delta] | \forall i \in I([\Delta] \in \|\delta_i\|^+)\},$$

and hence $\forall i \in I(\vdash_{\mathrm{cf}} \beta^\bullet \Rightarrow \delta_i$ and $\vdash_{\mathrm{cf}} \gamma^\bullet \Rightarrow \delta_i)$. Thus, we obtain $\forall i \in I(\vdash_{\mathrm{cf}} (\beta \vee \gamma)^\bullet \Rightarrow \delta_i)$ by ($\bullet\vee$left). This means $[(\beta \vee \gamma)^\bullet] \in \bigcap_{i \in I} \|\delta_i\|^+ = W$.

Second, we show $v^-(\beta \vee \gamma) \subseteq \|\beta \vee \gamma\|^-$. Suppose $[\Gamma] \in v^-(\beta \vee \gamma)$. Then, we have $[\Gamma] \in \mathrm{cl}(v^-(\beta) \cup v^-(\gamma))$, i.e.

$$\forall W[W \in D \text{ and } v^-(\beta) \cup v^-(\gamma) \subseteq W \text{ imply } [\Gamma] \in W].$$

We take $\|\beta \vee \gamma\|^-$ for W. If we can show $v^-(\beta) \cup v^-(\gamma) \subseteq \|\beta \vee \gamma\|^-$, then $[\Gamma] \in \|\beta \vee \gamma\|^-$. Thus, we prove this. Suppose $[\Delta] \in v^-(\beta) \cup v^-(\gamma)$. Then, $[\Delta] \in v^-(\beta) \cup v^-(\gamma) \subseteq \|\beta\|^- \cup \|\gamma\|^-$ by the induction hypothesis, and hence we obtain $[\Delta] \in \|\beta\|^-$ or $[\Delta] \in \|\gamma\|^-$, i.e. $\vdash_{\mathrm{cf}} \Delta \Rightarrow \beta^\bullet$ or $\vdash_{\mathrm{cf}} \Delta \Rightarrow \gamma^\bullet$. For both cases, we can obtain $\vdash_{\mathrm{cf}} \Delta \Rightarrow (\beta \vee \gamma)^\bullet$ by ($\bullet\vee$right1) or ($\bullet\vee$right2). This means $[\Delta] \in \|\beta \vee \gamma\|^-$. □

By using this key lemma, we can obtain the completeness theorem for BIQL as follows. If formula α is true, then $[] \in v^+(\alpha)$. On the other hand $v^+(\alpha) \subseteq \|\alpha\|^+$, and hence $[] \in \|\alpha\|^+$, that means "α is cut-free provable." By combining this with the soundness theorem, we also obtain the cut-elimination theorem for BIQL. Using a similar way, we can also prove the completeness and cut-elimination theorems for QIQL.

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